1 Limits

1.1 Limit Rules

Limit Theorems: Let $\lim_{x\to x_0} f(x) = L_1$ and $\lim_{x\to x_0} g(x) = L_2$. Then

1) $\lim_{x\to x_0} (f \pm g)(x) = \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x) = L_1 \pm L_2$

2) $\lim_{x\to x_0} (fg)(x) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x) = L_1 L_2$

3) $\lim_{x\to x_0} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \frac{L_1}{L_2}$, provided $L_2 \neq 0$

4) $\lim_{x\to x_0} (kf)(x) = k \cdot \lim_{x\to x_0} f(x) = kL_1$ for a real number k

5) $\lim_{x\to x_0} b = b$ for $b \in \mathbb{R}$

6) $\lim_{x\to x_0} x = x_0$

Limits of Polynomial Functions: For a polynomial $p(x)$, $\lim_{x\to x_0} p(x) = p(x_0)$.

Limits of Rational Functions: For a rational function $r(x) = \frac{N(x)}{D(x)}$ and x_0 in the domain of r, $\lim_{x\to x_0} r(x) = r(x_0) = \frac{N(x_0)}{D(x_0)}$.

Limits of Composite Functions: If $\lim_{x\to x_0} g(x) = L$ and $\lim_{x\to L} f(x) = f(L)$, then

$$\lim_{x\to x_0} (f \circ g)(x) = f\left(\lim_{x\to x_0} g(x)\right) = f(L).$$

Theorem: If x_0 is not an endpoint of the domain of f then

$$\lim_{x\to x_0} f(x) = L \iff \lim_{x\to x_0^-} f(x) = L \text{ and } \lim_{x\to x_0^+} f(x) = L.$$

Strategy for Limits at Infinity of Rational Functions:
Divide each term by the highest power of x (Case 1 and Case 2).

1) Degree of numerator and denominator are equal. Then divide the leading coefficients.

2) Degree of denominator is larger than degree of numerator. Then the limit is 0.

3) Degree of the numerator is larger than degree of the denominator. The limit will be $\pm\infty$.

Theorem: For two functions f and h that are equal for all values in an open interval containing x_0 (except possibly at x_0), if either function has limit L at x_0, then so does the other:

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = L.$$

Note that this is what permits us to cancel out common factors in a limit of a fraction.

Theorem:

1) If $\lim_{x\to x_0} f(x) = \infty$ and $\lim_{x\to x_0} g(x) = L > 0$, then $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = \infty$.

2) If $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = \pm\infty$, then $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = 0$.

3) If $\lim_{x\to x_0} f(x) = L \neq 0$ and $\lim_{x\to x_0} g(x) = 0$, then $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = \pm\infty$.

1.2 Practice

1) The Strategy for Limits at Infinity of Rational Functions states we can divide each term by the highest power of x when we are trying to calculate the limit as x approaches ∞. Think about why this works. Does it work for limits as x approaches $-\infty$? Is this helpful if we are calculating the limit as x approaches 2?

2) Find the limits of the following:

a) $\lim_{x\to 3} \dfrac{2x^2-5x-3}{x-3}$ b) $\lim_{x\to\infty} (x^3-4x^2+5)$ c) $\lim_{x\to-\infty} (x^3-4x^2+5)$

d) $\lim_{x\to 0} \left(\dfrac{5}{x^2+x} - \dfrac{5}{x}\right)$ e) $\lim_{x\to 1/2} \dfrac{x^{-1}-2}{x-1/2}$ f) $\lim_{x\to-\infty} \dfrac{2x^2}{x^2-4}$

g) $\lim_{x\to\infty} \dfrac{x^3}{4x^2+3}$ h) $\lim_{x\to 2} \dfrac{x^4-16}{x-2}$ i) $\lim_{x\to 0} |x-1|$

j) $\lim_{x\to -3} |x-1|$ k) $\lim_{x\to\infty} \dfrac{2x^3}{3x^2-4}$ l) $\lim_{x\to-\infty} \dfrac{x+2}{x^2+x+1}$

m) $\lim_{x\to-\infty} \dfrac{3x^3}{3x^2-2}$ n) $\lim_{x\to\infty} \dfrac{-3x^2}{4x+4}$ o) $\lim_{x\to\infty} \dfrac{x+1}{2x^2+2x+1}$

p) $\lim_{x\to 0} \sqrt{\cos(x)}$ q) $\lim_{x\to 1} e^{1/x}$ r) $\lim_{x\to\pi/2} \tan(x^2)$

s) $\lim_{x\to-\infty} \dfrac{5x^2+2}{3x^3-2x}$ t) $\lim_{x\to -2} \dfrac{\frac{1}{2}+\frac{1}{x}}{x+2}$ u) $\lim_{x\to -1} \dfrac{x^4+3x^3-x^2+x+4}{x+1}$

3) Find the limit of the piece-wise function:

a) $\lim_{x\to 2} f(x)$ for $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{for } x \neq 2 \\ 2 & \text{for } x = 2 \end{cases}$

b) $\lim_{x\to 2} f(x)$ for $f(x) = \begin{cases} x^2+5 & \text{for } x \leq 2 \\ -x+11 & \text{for } x > 2 \end{cases}$

c) $\lim_{x\to 1} f(x)$ for $f(x) = \begin{cases} \ln(x) & \text{for } 0 < x \leq 1 \\ \cos(x) & \text{for } x > 1 \end{cases}$

d) $\lim_{x\to -1^+} f(x)$ for $f(x) = \begin{cases} x+2 & \text{for } x \leq -1 \\ \dfrac{1}{x} & \text{for } x > -1 \end{cases}$

2 Asymptotes

Def: If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is a **horizontal asymptote** of the graph of f.

Def: If f increases without bound as x approaches x_0 from either the left or the right, then the vertical line $x = x_0$ is a **vertical asymptote** of the graph of f.

2.1 Practice

1) This question should help clarify the difference between vertical and horizontal asymptotes.

   a) Does a function ever cross its vertical asymptote?

   b) Does a function ever cross its horizontal asymptote? Hint: Consider the horizontal asymptote and graph of $f(x) = \dfrac{x+2}{x^2+1}$.

   c) Which type of asymptote corresponds to the domain of the function?

2) Find the vertical asymptote(s), if any, of the following:

   a) $f(x) = \dfrac{1}{x^2-1}$ b) $f(x) = \tan(x)$ c) $f(x) = \csc(x)$ d) $f(x) = \ln(x)$

3) Find the horizontal asymptote(s), if any, of the following:

   a) $f(x) = \sin(x)$ b) $f(x) = e^x$ c) $f(x) = \dfrac{x-1}{\sqrt{x}}$ d) $f(x) = \dfrac{3x^3+4}{x^3-x+2}$

3 Continuity

Def: A function f is **continuous at the point** x_0 if

1) f is defined at x_0, so $f(x_0)$ exists.
2) $\lim_{x \to x_0} f(x)$ exists.
3) $\lim_{x \to x_0} f(x) = f(x_0)$.

If a function is continuous at every point x_0 in its domain, we say it is continuous on its domain.

Continuity of Composite Functions: If g is continuous at x_0 and f is continuous at $g(x_0)$, then $f \circ g$ is continuous at x_0.

3.1 Practice

1) Determine if the following functions are continuous at the given point or on the given interval:

   a) $f(x) = \sqrt{x}$ at $x = -1$ and at $x = 4$

   b) $f(x) = \dfrac{x^2 - 4}{x - 2}$ at $x = 2$

   c) $f(x) = \begin{cases} \dfrac{x^2 - 1}{x + 1} & \text{for } x < -1 \\ x & \text{for } -1 \le x \le 2 \\ -x + 4 & \text{for } x > 2 \end{cases}$ at $x = -1$ and at $x = 2$

   d) $f(x) = e^x$ on the entire real line

   e) $f(x) = \ln(x)$ on the entire real line

   f) $f(x) = \ln(x^3)$ at $x = -1$

   g) $f(x) = \tan(\sqrt{x})$ at $x = \dfrac{\pi^2}{4}$

   h) $f(x) = \tan(\cos(x))$ on the entire real line

2) Find the a and/or b value such that $f(x)$ is continuous on the entire real line:

   a) $f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & \text{for } x \neq 3 \\ a & \text{for } x = 3 \end{cases}$

   b) $f(x) = \begin{cases} 2 & \text{for } x \le -1 \\ ax + b & \text{for } -1 < x < 3 \\ a - 2x & \text{for } x \ge 3 \end{cases}$

   c) $f(x) = \begin{cases} \cos(x) & \text{for } x \le \dfrac{\pi}{2} \\ \sin(x) + a & \text{for } x > \dfrac{\pi}{2} \end{cases}$

   d) $f(x) = \begin{cases} x^3 & \text{for } x \le 2 \\ ax^2 & \text{for } x > 2 \end{cases}$

4 Squeeze Theorem

Squeeze Theorem: Let I be an open interval containing the point x_0, and let functions f, g and h satisfy the inequalities $f(x) \leq g(x) \leq h(x)$ for all $x \in I$ except possibly at x_0. If

$$\lim_{x \to x_0} f(x) = L = \lim_{x \to x_0} h(x),$$

then

$$\lim_{x \to x_0} g(x) = L.$$

4.1 Practice

1) This problem is meant to walk you through how to approach Squeeze Theorem problems.

   a) Consider $\displaystyle\lim_{x \to 0^-} x^3 \cos\left(\frac{2}{x}\right)$.

   b) Which function can we easily bound, $\cos\left(\frac{2}{x}\right)$ or x^3? [Hint: Think of the graphs of the functions.]

   c) Using that
   $$-1 \leq \cos\left(\frac{2}{x}\right) \leq +1,$$
   how do we get an inequality for $x^3 \cos\left(\frac{2}{x}\right)$ [Hint: Multiply the inequality by x^3.]

   d) Since we are looking at the limit at $x \to 0^-$, we know $x < 0$, so $x^3 < 0$. That means the inequality signs switch directions, so we get
   $$-x^3 \leq x^3 \cos\left(\frac{2}{x}\right) \leq x^3.$$

   e) We know how to find the limit of x^3 and $-x^3$ as $x \to 0^-$, so now we apply the Squeeze Theorem to get our final answer.

2) Compute the following limits using the Squeeze Theorem:

   a) $\displaystyle\lim_{x \to \infty} \frac{\sin(x)}{x}$ b) $\displaystyle\lim_{x \to \infty} \frac{2 - \cos(x)}{x + 3}$ c) $\displaystyle\lim_{x \to \infty} \frac{\cos^2(2x)}{3 - 2x}$

5 Intermediate Value Theorem

Intermediate Value Theorem: Let f be continuous on the interval $[a, b]$ and let L be a number between $f(a)$ and $f(b)$. Then there exists at least one number c with $a < c < b$ such that

$$f(c) = L.$$

5.1 Practice

1) For each function, does the Intermediate Value Theorem apply for the given interval?

   a) $f(x) = 3x^2 + 2x - 1$ on $[0, 4]$

   b) $f(x) = \tan(x)$ on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

   c) $f(x) = \cos(x)$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

   d) $f(x) = \tan(x)$ on $[0, \pi]$

2) For each function, find the c value guaranteed by the Intermediate Value Theorem:

   a) $f(x) = 3x^2 + 2x - 1$ on $[0, 4]$, $f(c) = 15$

   b) $f(x) = \dfrac{x^2 + x}{x - 1}$ on $\left[\frac{5}{2}, 4\right]$, $f(c) = 6$

6 Instantaneous Velocity

Def: Let s be a continuous function and let $s(t)$ denote the position function for motion along a straight line, where t represents time with $a \leq t \leq b$. The instantaneous velocity at time t is denoted by the symbol $v(t)$. Then the instantaneous velocity at any time $t \in [a, b]$ is given by

$$v(t) = \lim_{h \to 0} \frac{s(t + h) - s(t)}{h}$$

provided the limit exists.

6.1 Practice

1) A ball is thrown vertically into the air from ground level with an initial velocity of 15 m/s. Its height at time t is $s(t) = 15t - 4.8t^2$. What is the instantaneous velocity at $t = 1$?

2) Let $p(t) = t^3 - 45t$ denote the distance (in meters) to the right of the origin of a particle at time t minutes after noon. Find the instantaneous velocity of the particle at 12:02 *pm*.

3) A particle is moving along a straight line so that its position at time t seconds is given by $s(t) = 4t^2 - t$, in meters. Find the instantaneous velocity at $t = 2$.

1 Derivatives

1.1 Definitions

Def: Let the function f be defined on the interval (a, b) and let $c \in (a, b)$. The **derivative of f at c** is given by

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

provided the limit exists. When $f'(c)$ exists, we say the function f is **differentiable at** c. If $f'(c)$ exists for each $c \in (a, b)$, we say f is **differentiable on** (a, b). The derivative is the instantaneous rate of change of f at c.

$$average\ rate\ of\ change = \frac{f(c+h) - f(c)}{h}$$

Def: Let the function f be defined on the interval $[a, b)$. The **right-hand derivative of f at a** is the number $f'(a^+)$ given by

$$f'(a^+) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists. When $f'(a^+)$ exists, we say the function f is **differentiable at $x = a$ from the right**. Similarly, if f is defined on $(a, b]$, then the **left-hand derivative of f at b** is the number $f'(b^-)$ given by

$$f'(b^-) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$$

provided the limit exists. When $f'(b^-)$ exists, we say the function f is **differentiable at $x = b$ from the left**.

Def: For f defined on (a, b), we say that f is **differentiable on the open interval** (a, b) if it is differentiable at each point in (a, b). When this is true, we use the symbols $f'(x)$ and $\frac{df}{dx}$ to denote the function whose value at each $c \in (a, b)$ is equal to $f'(c)$ at that point. When the domain of f is a closed and bounded interval $[a, b]$, f is differentiable on (a, b), and the left- and right-hand derivatives $\frac{df}{dx}(a^+)$ and $\frac{df}{dx}(b^-)$ exist, then we say f is **differentiable on the closed interval** $[a, b]$.

Position, Velocity and Acceleration: Given a position function $s(t)$ for an object at time t with velocity $v(t)$ and acceleration $a(t)$,

$$v(t) = s'(t), \qquad a(t) = v'(t) = s''(t)$$

1.2 Rules

Theorem: Let the function f be defined on an interval (a, b) and let f be differentiable at the point $c \in (a, b)$. Then f is continuous at c.

Summary:

1) Differentiable at a point $x = c$ implies continuous at $x = c$.

2) Not continuous at a point $x = c$ implies not differentiable at $x = c$.

3) Continuous at a point $x = c$ doesn't say if differentiable or not (tells us nothing).

Note: If f has derivative $f'(c)$ at $x = c$, then $f'(c^+) = f'(c) = f'(c^-)$. On the other hand, if $f'(c^+) \neq f'(c^-)$, then $f'(c)$ cannot exist.

Derivative of a Sum: Let f and g be functions that are differentiable on a common interval I. Then $f + g$ is differentiable on I, and

$$(f + g)'(x) = f'(x) + g'(x) \text{ for all } x \in I.$$

Constant Multiple Rule for Derivatives: If f is differentiable on some interval I and k is any constant, then kf is differentiable on I. The derivative of $(kf)(x)$ on I is given by

$$(kf)'(x) = kf'(x).$$

Generalized Power Rule: Let p be any nonzero real number. Then

$$\frac{d}{dx}(x^p) = p \cdot x^{p-1}$$

provided x is in the domain of x^p and x^{p-1}.

Derivative of a Polynomial: Polynomial functions are differentiable on the entire real line. In particular, the derivative of the n^{th}-degree polynomial function $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, for $n \geq 1$, is the polynomial

$$P'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + 2a_2x + a_1.$$

Product Rule: Let f and g be differentiable on a common interval I. Then fg is differentiable on I, and

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Quotient Rule:. Let f and g be functions that are differentiable on a common interval I. If $g(x) \neq 0$, then $\dfrac{f}{g}$ is also differentiable on I and

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

Mnemonic device: Low D-High - High D-Low over Low-Squared.

Corollary: If f is differentiable and never zero on an interval I, then $\dfrac{1}{f}$ is differentiable on I and

$$\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{(f(x))^2}.$$

Chain Rule: Let the function g be differentiable at a point $x = a$, and suppose that the function f is defined on an open interval containing the point $g(a)$ and differentiable at $g(a)$. Then the composite function $f \circ g$ is differentiable at $x = a$ and

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

Theorem: If g is a differentiable function on an interval I, then at each point $x \in I$ and for each real number p,

$$\frac{d}{dx}(g(x))^p = \frac{d}{dx}g^p(x) = pg^{p-1}(x) \cdot g'(x),$$

provided $g^p(x)$ and $g^{p-1}(x)$ are defined.

Fact:

a) $\displaystyle\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1,$

b) $\displaystyle\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta} = 0$

Derivatives of Trig Functions:

a) $\dfrac{d}{d\theta}\sin(\theta) = \cos(\theta)$

b) $\dfrac{d}{d\theta}\cos(\theta) = -\sin(\theta)$

c) $\dfrac{d}{d\theta}\tan(\theta) = \sec^2(\theta)$

d) $\dfrac{d}{d\theta}\cot(\theta) = -\csc^2(\theta)$

e) $\dfrac{d}{d\theta}\sec(\theta) = \sec(\theta)\tan(\theta)$

f) $\dfrac{d}{d\theta}\csc(\theta) = -\csc(\theta)\cot(\theta)$

Note: $\sin(\theta)$ and $\cos(\theta)$ are differentiable for all real numbers. The other four trig functions are differentiable on their respective domains.

Derivatives of the Inverse Trig Functions:

a) $\dfrac{d}{dx}\arcsin(x) = \dfrac{1}{\sqrt{1-x^2}}$, for $|x| < 1$

b) $\dfrac{d}{dx}\arccos(x) = -\dfrac{1}{\sqrt{1-x^2}}$, for $|x| < 1$

c) $\dfrac{d}{dx}\arctan(x) = \dfrac{1}{1+x^2}$, for $x \in \mathbb{R}$

d) $\dfrac{d}{dx}\operatorname{arccot}(x) = -\dfrac{1}{1+x^2}$, for $x \in \mathbb{R}$

e) $\dfrac{d}{dx}\operatorname{arcsec}(x) = \dfrac{1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$

f) $\dfrac{d}{dx}\operatorname{arccsc}(x) = -\dfrac{1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$

Derivatives of Natural Exponential Functions:

$$\frac{d}{dx}(e^x) = e^x, \qquad \frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)} \cdot f'(x)$$

Derivatives of Other Exponential Functions: For each $a > 0$,

$$\frac{d}{dx}(a^x) = a^x \ln(a), \qquad \frac{d}{dx}\left(a^{f(x)}\right) = a^{f(x)} \ln(a) f'(x)$$

Derivatives of Logarithmic Functions: For each $a > 0$ with $a \neq 1$ and $x > 0$,

$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}, \qquad \frac{d}{dx}\log_a(f(x)) = \frac{f'(x)}{f(x)\ln(a)}$$

Derivatives of Natural Logarithmic Functions:

$$\frac{d}{dx}\ln(x) = \frac{1}{x}, \qquad \frac{d}{dx}\ln\left(f(x)\right) = \frac{f'(x)}{f(x)}$$

1.3 Practice

1) Use the definition of the derivative to differentiate:

a) $f(x) = x^2 + 7$

b) $f(x) = \sqrt{x-1}$

c) $f(x) = ax^2 + c$

d) $f(x) = x^2 - 5x + 2$

e) $f(x) = \dfrac{1}{x}$

f) $f(x) = \dfrac{1}{x-2}$

g) $f(x) = \sqrt{x^2 - 1}$

h) $f(x) = ax^3$

2) Use any method/rule to find the derivative:

a) $f(x) = e^x \sin(x)$

b) $f(x) = (x^4 + 3x)^{-1}$

c) $f(x) = 3x^2(x^3+1)^7$

d) $f(x) = \cos^4(x) - 2x^2$

e) $f(x) = \dfrac{x}{1+x^2}$

f) $f(x) = \dfrac{x^2 - 1}{x}$

g) $f(x) = 3x^2\sqrt{x}$

h) $f(x) = \ln(xe^{7x})$

i) $f(x) = \dfrac{2x^4 + 3x^2 - 1}{x^2}$

j) $f(x) = \dfrac{4(3x-1)^2}{x^2 + 7^x}$

k) $f(x) = 2x - \dfrac{4}{\sqrt{x}}$

l) $f(x) = \dfrac{x}{\sqrt{1 - (\ln(x))^2}}$

m) $f(x) = x^3 \sqrt[5]{7-x}$

n) $f(x) = (xe^x)^\pi$

o) $f(x) = [\arctan(2x)]^{10}$

p) $f(x) = (e^{2x} + e)^{\frac{1}{2}}$

q) $f(x) = (7x + \sqrt{x^2+3})^6$

r) $f(x) = e^x(x^2+3)(x^3+4)$

s) $f(x) = \sqrt{\dfrac{2x+5}{7x-9}}$

t) $f(x) = \dfrac{\frac{1}{x} + \frac{1}{x^2}}{x-1}$

u) $f(x) = \left[\ln(5x^2+9)\right]^3$

v) $f(x) = \sec(x)\sin(3x)$

w) $f(x) = \tan(\cos(x))$

x) $f(x) = \log_5(3x^2 + 4x)$

y) $f(x) = e^{5\tan(x)}$

z) $f(x) = \arcsin(x^3 + 2)$

3) Find the equation of the tangent line to $f(x) = \dfrac{1}{2}x^5 - 2x$ at

a) $x = 0$

b) $x = 1$

c) $x = 2$

4) Find the equation of the tangent line to $f(x) = e^{2x^3}$ at

a) $x = 0$

b) $x = 1$

c) $x = -2$

5) Suppose the position of an object is given by $s(t) = x^4 - 3x^2 - x + 1$ in m, where time is measured in s.

   a) Find the velocity $v(t)$ and $v(2)$.
   b) Find the acceleration $a(t)$ and $a(2)$.

6) Suppose the height of a coin dropped off a 100 ft tall bridge is given by $s(t) = 100 - t^2$ in ft, where time is measured in s.

   a) How long does it take for the coin to hit the water?
   b) Find the velocity $v(t)$.
   c) What is the coin's velocity when it hits the water?

2 Derivative Techniques

Method of Implicit Differentiation:

Let $y = f(x)$. Consider $F(x, y) = G(x, y)$. Carry out the following steps to compute $\frac{dy}{dx}$.

1) Use derivative rules to compute the derivative of both sides of the equation, using $\frac{d}{dx}(x) = 1$ (as usual) and $\frac{d}{dx}(y) = \frac{dy}{dx}$.

2) Solve the resulting equation for $\frac{dy}{dx}$, if possible.

Logarithmic Differentiation Procedure:

To compute the derivative of $y = f(x)$ using the method of logarithmic differentiation, proceed as follows:

1) Take the natural log of both sides of the equation $y = f(x)$ and use the properties of $\ln(x)$ to expand $\ln(f(x))$.

2) Use implicit differentiation with respect to x to compute the derivative of both sides of your equation.

3) Solve for $\frac{dy}{dx}$.

Recall:

$$\ln(ab) = \ln(a) + \ln(b)$$

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$$

$$a \cdot \ln(b) = \ln(b^a)$$

2.1 Practice

1) Differentiate the following using implicit and/or logarithmic differentiation.

a) $3y = xe^{5y}$

b) $xy + y^2 + x^3 = 7$

c) $\dfrac{\sin(y)}{y^2+1} = 3x$

d) $2x^3 = 2y^2 + 5$

e) $3x^2 + 3y^2 = 2$

f) $5y^2 = 2x^3 - 5y$

g) $4x^2 = 2y^3 + 4y$

h) $3x^2y^2 = 4x^2 - 4xy$

i) $5x^3 + xy^2 = 5x^3y^3$

j) $2x^3 = (3xy+1)^2$

k) $x^2 = (4x^2y^3+1)^2$

l) $\sin(2x^2y^3) = 3x^3 + 1$

m) $3x^2 + 3 = \ln(5xy^2)$

n) $y = 2x^{2x}$

o) $y = 5x^{5x}$

p) $y = 4x^{x^4}$

q) $y = (3x^4+4)^3\sqrt{5x^3+1}$

r) $y = \dfrac{(3x^4-2)^5}{(3x^3+4)^2}$

s) $y = \sqrt{3x^2+1}(3x^4+1)^3$

t) $y = \dfrac{\sqrt{2x^3+3}}{(x^4-3)^3}$

u) $y = (2x^2-5)^3\sqrt{x^2-2}$

v) $y = \dfrac{(5x-4)^4}{(3x^2+5)^5\ (5x^4-3)^3}$

2) Find the equation of the tangent line to $x^2y^2 - 2x = 4 - y$ at the point $(2, -2)$.

3) Find all points where the tangent line to $x^2 + y^3 - 3y = 4$ is horizontal and where it is vertical.

3 Derivative Applications

Related Rates Strategy:

1) Identify all given quantities as well as quantities to be determined and label them.

2) Write an equation involving the variables whose rates of change are given or to be determined (relate the variables).

3) Using implicit differentiation and the Chain Rule, differentiate both sides with respect to t.

4) Substitute into the resulting equation all known values for variables and their rates of change, then solve for the unknown.

3.1 Practice

1) Water leaking onto a floor forms a circular pool. The radius of the pool increases at a rate of 4 cm/min. How fast is the area of the pool increasing when the radius is 5 cm?

2) Oil spilling from a ruptured tanker spreads in a circle on the surface of the ocean. The area of the spill increases at a rate of 9π m^2/min. How fast is the radius of the spill increasing when the radius is 10 m?

3) A conical paper cup is 10 cm tall with a radius of 10 cm. The cup is being filled with water so that the water level rises at a rate of 2 cm/sec. At what rate is the water being poured into the cup when the water level is 8 cm?

4) A spherical balloon is inflated so that its radius, r, increases at a rate of $\frac{2}{r}$ cm/sec. How fast is the volume of the balloon increasing when the radius is 4 cm?

5) An observer stands 700 ft away from a launch pad to observe a rocket launch. The rocket blasts off and maintains a velocity of 900 ft/sec. Assume the scenario can be modeled as a right triangle. How fast is the observer to rocket distance changing when the rocket is 2400 ft from the ground?

6) An 8 ft ladder is leaning against a wall. The top of the ladder is sliding down the wall at the rate of 2 ft/sec. How fast is the bottom of the ladder moving along the ground at the point in time when the bottom of the ladder is 4 ft from the wall?

7) The width of a rectangle is increasing at a rate of 2 cm/sec, while the length increases at 3 cm/sec. At what rate is the area increasing when $w = 4$ cm and $l = 5$ cm?

1 Linearization

Def: Let f be differentiable at $c \in (a, b)$. The **linearization of f at $x = c$** is the function which is defined by the equation of the tangent line to f at $x = c$:

$$f(x) \approx L_c(x) = f(c) + f'(c)(x - c)$$

The value $L_c(x)$ is called the **linear approximation of f at values of x near c.**

1.1 Practice

1) Use linearization to approximate $y = \sqrt{4 + \sin(x)}$ at $x = 0.12$, using the tangent line at $x = 0$.
2) Use linearization to approximate $\sin(.2)$. What is an appropriate c value to use?
3) Use linearization to approximate $f(5.02)$ if $f(x) = \dfrac{1}{\sqrt{4 + x}}$.
4) Use linearization to approximate the value of $(2.003)^4$.

2 Newton's Method

Newton's Method: Let f be a differentiable function on an interval (a, b) containing a root (or zero) x^* of f. That is, $f(x^*) = 0$. Let $x_1 \in (a, b)$. Newton's method to approximate the root x^* is the iterative process

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

If x_{n+1} is outside the interval (a, b), the process stops and the method fails. If x_{n+1} remains in the interval (a, b), we adopt the 10-digit rule and stop the process when x_{n+1} agrees with x_n to 10 digits (or some other degree of "matching").

2.1 Practice

1) Use Newton's Method to find a solution to $x^2 - 17 = 0$.
2) Find the solution to $\cos(x) = x$ (make a sketch to help with your initial guess).
3) Use Newton's Method to find $\ln(2)$. (Hint: Start by finding an equation whose solution is $\ln(2)$). What happens if you start with the initial guess -4?
4) Use Newton's method to approximate $\sqrt[100]{100}$ to four decimal places.
5) Use Newton's method to find the roots of $\dfrac{1}{x} = 1 + x^3$ to 3 decimal places.
6) The equation $x^2 = 2^x$ has two integer solutions $x = 2$ and $x = 4$. Use Newton's method to approximate the other solution. Graph the function to inspire your first estimate.

3 Mean Value Theorem

Mean Value Theorem: Let the function f be continuous on the closed and bounded interval $[a, b]$ and differentiable on the open interval (a, b). Then there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Rolle's Theorem: Let f be a function that is continuous on the closed, bounded interval $[a, b]$, differentiable on the open interval (a, b), and suppose that $f(a) = f(b)$. Then there is at least one point $c \in (a, b)$ where $f'(c) = 0$.

3.1 Practice

1) For each problem, find the values of c that satisfy the Mean Value Theorem.

   a) $y = -x^2 + 8x - 17; \quad [3, 6]$

   b) $y = x^3 - 9x^2 + 24x - 18; \quad [2, 4]$

   c) $y = -\frac{x^2}{2} + x - \frac{1}{2}; \quad [-2, 1]$

   d) $y = \frac{x^2}{2} - 2x - 1; \quad [-1, 1]$

   e) $y = x^3 + 3x^2 - 2; \quad [-2, 0]$

   f) $y = \frac{x^2}{2x - 4}; \quad [-4, 1]$

2) For each problem, determine if the Mean Value Theorem can be applied. If it can, find all values of c that satisfy the theorem. If it cannot, explain why.

   a) $y = -\frac{x^2}{4x + 8}; \quad [-3, -1]$

   b) $y = -(6x + 24)^{2/3}; \quad [-4, -1]$

   c) $y = \frac{-x^2 + 9}{4x}; \quad [1, 3]$

   d) $y = (x - 3)^{2/3}; \quad [1, 4]$

4 Extrema

Def: Let the function f be defined on a domain $\mathcal{D}$, and let $c \in \mathcal{D}$. Then we say $f(c)$ is the **absolute (global) maximum** value, or simply the **maximum** of the function, if

$$f(x) \leq f(c)$$

for all $x \in \mathcal{D}$. We say that f has an absolute maximum at $x = c$. Similarly, we say $f(c)$ is the **absolute (global) minimum** value, or simply the **minimum** of the function, if

$$f(x) \geq f(c)$$

for all $x \in \mathcal{D}$. We say that f has an absolute minimum at $x = c$.

Def: Let f be a function with domain $\mathcal{D}$ and let $c \in \mathcal{D}$. Then we say $f(c)$ is a **local maximum** value of f if

$$f(x) \leq f(c)$$

for all $x \in \mathcal{D}$ *near* c. Similarly, we say that $f(c)$ is a **local minimum** value of f if

$$f(x) \geq f(c)$$

for all $x \in \mathcal{D}$ *near* c. Here, ”near” means some small interval around c.

Note: The maximum or minimum is $f(c)$. It occurs at $x = c$.

Def: A number $x = c$ in the domain of a continuous function f is called a **critical number** of f if either $f'(c) = 0$ or $f'(c)$ does not exist.

Extreme Value Theorem: Let the function f be continuous on the closed and bounded interval $[a, b]$. Then there are points $x_1, x_2 \in [a, b]$ such that $f(x_1)$ and $f(x_2)$ are absolute maximum and absolute minimum values of f on $[a, b]$, respectively. That is, for all $x \in [a, b]$,

$$f(x_2) \leq f(x) \leq f(x_1).$$

Procedure to Find Absolute Max/Min of f on $[a, b]$:

For f continuous on the closed and bounded interval $[a, b]$:

1) Find the critical numbers x_1, x_2, $\ldots$ of f. Ignore any critical numbers not in $[a, b]$. Create the set of y-values $S = \{f(x_1), f(x_2), \ldots\}$.

2) Add the values $f(a)$ and $f(b)$ to the set S: $S = \{f(a), f(x_1), f(x_2), \ldots, f(b)\}$.

3) The largest number in the set S is the absolute maximum of f on $[a, b]$, and the smallest number in the set S is the absolute minimum of f on $[a, b]$.

Def: Let the function f be defined on an interval I. f is **increasing** on I if, for $x_1, x_2 \in I$,

$$f(x_1) < f(x_2)$$

whenever $x_1 < x_2$. Similarly, f is **decreasing** on I if, for $x_1, x_2 \in I$,

$$f(x_1) > f(x_2)$$

whenever $x_1 < x_2$.

Theorem: Let the function f be differentiable on the open interval (a, b). Then

1) If $f'(x) > 0$ on (a, b), then f is increasing on (a, b).
2) If $f'(x) < 0$ on (a, b), then f is decreasing on (a, b).

First Derivative Test for Local Extrema: Suppose $c \in (a, b)$ is a critical number of f. Let f be continuous on (a, b), and differentiable on (a, b), except possibly at c.

1) If $f'(x) < 0$ for $x \in (a, c)$ and $f'(x) > 0$ for $x \in (c, b)$, then $f(c)$ is a local minimum value of f.
2) If $f'(x) > 0$ for $x \in (a, c)$ and $f'(x) < 0$ for $x \in (c, b)$, then $f(c)$ is a local maximum value of f.
3) If $f'(x)$ has the same sign on both (a, c) and (c, b), then f does not have a local extremum at c.

Def: Let the function f be differentiable on the interval (a, b).

1) If f' is increasing on (a, b), then the graph of f is **concave up** on (a, b).
2) If f' is decreasing on (a, b), then the graph of f is **concave down** on (a, b).

Theorem: Let f be defined and twice differentiable on an open interval (a, b). Then

1) If $f''(x) > 0$ on (a, b), then the graph of f is concave up on (a, b).
2) If $f''(x) < 0$ on (a, b), then the graph of f is concave down on (a, b).

Second Derivative Test for Local Extrema: Let $c \in (a, b)$ be a critical number of the function f, and assume that f'' is continuous on the open interval (a, b).

1) If $f''(c) > 0$, then $f(c)$ is a local minimum of f.
2) If $f''(c) < 0$, then $f(c)$ is a local maximum of f.
3) If $f''(c) = 0$, then this test gives no information about $f(c)$.

Def: Let f be continuous on an open interval (a, b) containing the point c, and differentiable on (a, b), except possibly at $x = c$. Then $(c, f(c))$ is a **point of inflection** if the graph of f changes concavity at $x = c$.

4.1 Practice

1) For each problem, find all relative maxima and minima, intervals where y is increasing/decreasing, and intervals where y is concave up/down.

   a) $y = -x^3 - 3x^2 - 1$

   b) $y = x^4 - 2x^2 + 3$

   c) $y = x^3 - 6x^2 + 9x + 1$

   d) $y = \dfrac{1}{x^3 - 3x^2}$

2) For each problem, find the absolute maxima and minima on the given closed interval.

   a) $y = -x^3 - 6x^2 - 9x + 3; \quad [-3, -1]$

   b) $y = \dfrac{8}{x^2 + 4}; \quad [0, 5]$

   c) $y = \dfrac{x^2}{3x - 6}; \quad [3, 6]$

   d) $y = (5x + 25)^{1/3}; \quad [-2, 2]$

   e) $y = x^3 - 3x^2 + 6; \quad [0, \infty)$

   f) $y = \dfrac{4}{x^2 + 2}; \quad (-5, -2]$

   g) $y = x^3 - 3x^2 - 3; \quad (0, 3)$

   h) $y = x^4 - 2x^2 - 3; \quad (0, \infty)$

5 Optimization

Basic Strategy for Optimization Problems:

1) Read the problem carefully and make a diagram, if needed.

2) List quantities that vary and those that are fixed.

3) Find constraint equation(s), if any.

4) Create equation related to quantity being optimized. Use constraint equation(s) to reduce problem to one variable.

5) Find local maxima or minima relevant to the problem.

6) Use extrema to answer additional questions, if needed.

5.1 Practice

1) The sum of two positive numbers is 48. What is the smallest possible value for the sum of their squares?

2) A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs $10 per square meter. Material for the sides costs $6 per square meter. Find the costs of materials for the cheapest such container.

3) A farmer is planting a small orchard. He is advised that, if he plants up to 60 trees of a particular type on his plot of land the average harvest from each tree will be about 120 kg, but for each additional tree planted the expected yield will go down by an average of 2 kg per tree as a result of overcrowding. How many trees should he plant for the maximum yield of fruit?

4) Suppose you had to use exactly 200 m of fencing to make either one square enclosure or two separate square enclosures of any size you wished. What plan would give you the least area? What plan would give you the greatest area?

5) You want to enclose two adjacent rectangular pens using 200 ft of fencing. What dimensions should be used to maximize the enclosed area?

6) Find two positive numbers such that their product is 192 and the sum of the first plus three times the second is minimized.

7) A wire 6 m long is cut into twelve pieces. These pieces are welded together at right angles to form the frame of a box with a square base. Where should the cuts be made to maximize the volume of the box? Where should the cuts be made to maximize the total surface area for the box?

8) Four ft of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?

9) The combined perimeter of a circle and a square is 16 in. Find the dimensions of the circle and square that produce a minimum total area.

10) Find two positive numbers such that the sum of the first and twice the second is 100 and their product is a maximum.

6 Indeterminate Forms

Indeterminate Forms: $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, 0^0, 1^\infty, \infty^0, \infty - \infty$.

l'Hopital's Rule for Indeterminate Forms of Type $\frac{0}{0}$:
Suppose that functions f and g are differentiable on an open interval containing $x = a$, that $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, and that $g'(x) \neq 0$ for all x near a, except possibly at $x = a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right hand side exists. Moreover, the limits may be left-handed or right-handed limits at $x = a$, and we may also have $a = \pm\infty$.

l'Hopital's Rule for Indeterminate Forms of Type $\frac{\infty}{\infty}$:
Suppose that functions f and g are differentiable on an open interval containing $x = a$, that $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty$, and that $g'(x) \neq 0$ for all x near a, except possibly at $x = a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right hand side exists. Moreover, the limits may be left-handed or right-handed limits at $x = a$, and we may also have $a = \pm\infty$.

For the remaining indeterminate forms, we can't apply l'Hopital's Rule directly. We instead try to convert the limit to one that gives the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

For example, if $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$, then

$$\lim_{x \to a} f(x)g(x)$$

is indeterminate, of the form $0 \cdot \infty$. However, we can rewrite it as

$$\lim_{x \to a} f(x)g(x) = \lim_{x \to a} \frac{g(x)}{\frac{1}{f(x)}} \quad \text{or} \quad \lim_{x \to a} f(x)g(x) = \lim_{x \to a} \frac{f(x)}{\frac{1}{g(x)}}.$$

This new limit is indeterminate of the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$, so we can use l'Hopital's Rule.

The indeterminate forms of type $1^\infty, \infty^0, 0^0$ can arise from limits of powers of the form

$$\lim_{x \to a} (f(x))^{g(x)}.$$

Convert the limit to a form we know how to deal with as follows:

$$\begin{aligned}\lim_{x\to a}(f(x))^{g(x)} &= \lim_{x\to a} e^{\ln\left(f(x)^{g(x)}\right)} && (e \text{ and } \ln \text{ are inverses})\\ &= \lim_{x\to a} e^{g(x)\ln(f(x))} && (\text{using log rules})\\ &= e^{\lim\limits_{x\to a} g(x)\ln(f(x))} && (\text{using limit rule for function composition})\end{aligned}$$

There is no set rule for indeterminate forms of type $\infty - \infty$.

6.1 Practice

1) Find the following limits:

a) $\displaystyle\lim_{x\to\infty}\frac{\ln(x)}{x-1}$ b) $\displaystyle\lim_{x\to 1}\frac{x^2-1}{x^2-x}$ c) $\displaystyle\lim_{x\to\infty}\frac{e^{x/10}}{x^3}$

d) $\displaystyle\lim_{x\to\pi^-}\frac{\sin(x)}{1-\cos(x)}$ e) $\displaystyle\lim_{x\to-\infty} x^2e^x$ f) $\displaystyle\lim_{x\to 0^+}(\csc(x)-\cot(x))$

g) $\displaystyle\lim_{x\to 0^+} x^x$ h) $\displaystyle\lim_{x\to 0}\frac{\sin(4x)}{\tan(5x)}$ i) $\displaystyle\lim_{x\to 1}\frac{\ln(x)}{\sin(\pi x)}$

j) $\displaystyle\lim_{x\to\infty}\sqrt{x^2+x}-x$ k) $\displaystyle\lim_{x\to\infty} x^{1/x}$ l) $\displaystyle\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right)$

m) $\displaystyle\lim_{x\to 0^+}\frac{2x+7}{4x+1}$ n) $\displaystyle\lim_{x\to 0^+}\left(\cot(x)-\frac{1}{x}\right)$ o) $\displaystyle\lim_{x\to 1}\frac{5\ln(x)}{x-1}$

p) $\displaystyle\lim_{x\to\infty}\frac{e^x}{e^x+e^{-x}}$ q) $\displaystyle\lim_{x\to 0}\frac{3x}{\ln(x+1)}$ r) $\displaystyle\lim_{x\to 0^+} 5x^2\ln(x)$

s) $\displaystyle\lim_{x\to\infty} 4xe^{-x}$ t) $\displaystyle\lim_{x\to\frac{\pi}{2}}(3\sec(x)-3\tan(x))$ u) $\displaystyle\lim_{x\to\infty}\left(\frac{x^2}{x-1}-\frac{x^2}{x+1}\right)$

v) $\displaystyle\lim_{x\to 0^+} 5\,(\tan(x))^{\sin(x)}$ w) $\displaystyle\lim_{x\to 0^+} 3x^x$ x) $\displaystyle\lim_{x\to 0}\frac{\sin(x^3)}{\sin^3(x)}$

2) Explain why l'Hopital's Rule will fail in the following two cases:

a) $\displaystyle\lim_{x\to 0}\frac{\sqrt{4x^2+3}}{x+3}$

Note that this is not an indeterminate form. Compare the correct limit with the limit you get if you apply l'Hopital's Rule.

b) $\displaystyle\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}$

(Hint: divide each term by e^x.)

7 Differentials

Theorem: If f is differentiable at c, then $\Delta y = f(c+\Delta x) - f(c)$ satisfies

$$\Delta y = (f'(c) + \varepsilon)\Delta x = f'(c)\Delta x + \varepsilon \Delta x$$

where the function ε has the property that

$$\lim_{\Delta x \to 0} \varepsilon(\Delta x) = 0.$$

We call $f'(c)\Delta x$ the **principal part** of Δy and $\varepsilon \Delta x$ the **error part** of Δy. The product $\varepsilon \Delta x$ is the error in using $f'(c)\Delta x$ to approximate Δy.

Def: For a differentiable function $y = f(x)$, the **differential** of y or **differential** of f is

$$dy = df = f'(x)\ dx,$$

where dx, the differential of x, is an independent variable associated with x (think Δx).

Def: The **relative change** of f as x changes from c to $c + \Delta x$ is

$$\frac{\Delta f}{f(c)} = \frac{\Delta y}{f(c)}.$$

Since df is an approximation of Δf, we can use $\dfrac{df}{f(c)}$ to approximate relative change.

7.1 Practice

1) Let $f(x) = 4\sin(x)$. Find the differential of f and approximate the change in f as x changes from $\frac{\pi}{6}$ to $\frac{\pi}{6} + 0.1$. Use this to approximate $f\left(\frac{\pi}{6} + 0.1\right)$. Now use the differential to approximate the relative change as x changes from $\frac{\pi}{6}$ to $\frac{\pi}{6} + 0.1$.

2) Use differentials to approximate the increase in surface area of a cube if the length of each edge is changed from 10 cm to 10.1 cm. What is the exact change in surface area?

3) Find the length of string you would need to wrap the string around the earth at the equator. Use 3960 mi as the radius of the earth. Use differentials to approximate how much extra string you would need if the radius were off by 1 ft.

8 Antiderivatives

Def: A function F is called an **antiderivative** of a function f on an interval I if $F'(x) = f(x)$ for all $x \in I$. The process of finding F given f is called antidifferentiation. The collection of functions $\{F(x) + C : C \in \mathbb{R}\}$ is the **family of antiderivatives** of f.

Def: If f is a function defined on an interval I, then the symbol

$$\int f(x)\ dx$$

is called the **indefinite integral** of the function f. It denotes the family of antiderivatives of f on the interval I. If $F(x)$ is a particular antiderivative of f, we use the notation

$$\int f(x)\ dx = F(x) + C$$

to denote the family of antiderivatives, where C is the **constant of integration**.

Properties of Indefinite Integrals: Let F and G be antiderivatives of f and g, respectively, and let a and b be constants. Then

1) $\displaystyle\int af(x)\ dx = a\int f(x)\ dx = aF(x) + C$

2) $\displaystyle\int (f(x) + g(x))\ dx = \int f(x)\ dx + \int g(x)\ dx = F(x) + G(x) + C$

3) $\displaystyle\int (af(x) + bg(x))\ dx = a\int f(x)\ dx + b\int g(x)\ dx = aF(x) + bG(x) + C$

Theorem: (Power Rule for Indefinite Integrals) The indefinite integral of the function $f(x) = x^p$ is the family of antiderivatives given by

$$\int x^p\ dx = \frac{x^{p+1}}{p+1} + C,\ C, p \in \mathbb{R},\ p \neq -1.$$

Derivative and Indefinite Integral Formulas for e^x and $\ln x$.

$\frac{d}{dx}F(x) = f(x)$	$\int f(x)\,dx = F(x) + C$
1. $\frac{d}{dx}e^x = e^x$	$\int e^x\,dx = e^x + C$
2. $\frac{d}{dx}\ln\lvert x\rvert = \frac{1}{x}$, $\lvert x\rvert \neq 0$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$

Derivative and Indefinite Integral Formulas for the Trigonometric Functions

$\frac{d}{dx}F(x) = f(x)$	$\int f(x)\,dx = F(x) + C$	
1. $\frac{d}{dx}\sin ax = a\cos ax$	$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$	$a \neq 0$
2. $\frac{d}{dx}\cos ax = -a\sin ax$	$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$	$a \neq 0$
3. $\frac{d}{dx}\tan ax = a\sec^2 ax$	$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$	$a \neq 0$
4. $\frac{d}{dx}\cot ax = -a\csc^2 ax$	$\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C$	$a \neq 0$
5. $\frac{d}{dx}\sec ax = a\sec ax\tan ax$	$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax + C$	$a \neq 0$
6. $\frac{d}{dx}\csc ax = -\csc ax\cot ax$	$\int \csc ax\cot ax\,dx = -\frac{1}{a}\csc ax + C$	$a \neq 0$

Derivative and Indefinite Integral Formulas for the Inverse Trigonometric Functions

$\frac{d}{dx}F(x) = f(x)$		$\int f(x)\,dx = F(x) + C$
1. $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$,	$\lvert x\rvert < 1$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$
2. $\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$,	$\lvert x\rvert < 1$	$\int -\frac{1}{\sqrt{1-x^2}}\,dx = \arccos x + C$
3. $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$,	$x \in \mathbb{R}$	$\int \frac{1}{1+x^2}\,dx = \arctan x + C$
4. $\frac{d}{dx}\text{arcsec}\, x = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$,	$\lvert x\rvert > 1$	$\int \frac{1}{\lvert x\rvert\sqrt{x^2-1}}\,dx = \text{arcsec}\, x + C$

8.1 Practice

1) Find the following antiderivatives:

a) $\int (4x^3 - 12x^2 + 6x - 1)\ dx$

b) $\int (x^5 + 3x^4 - x^2 + 4)\ dx$

c) $\int 7\sin(x)\ dx$

d) $\int -3\csc(x)\cot(x)\ dx$

e) $\int (4e - 7x)\ dx$

f) $\int\ d\theta$

g) $\int \frac{3}{x}\ dx$

h) $\int \frac{2}{\sec(x)}\ dx$

i) $\int y^2\sqrt[3]{y}\ dy$

j) $\int \left(5\cos(x) - \sec^2(3x)\right)\ dx$

k) $\int \left(5x^6 + \csc(x)\cot(x)\right)\ dx$

l) $\int -\frac{5}{\sqrt{1-x^2}}\ dx$

m) $\int 10e^x\ dx$

n) $\int -\frac{6}{5x}\ dx$

o) $\int \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right)\ dx$

p) $\int \frac{9}{1+x^2}\ dx$

1 Areas and Riemann Sums

Summation Notation:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_{n-1} + a_n$$

The capital Greek letter Σ is the symbol for sum and the symbol a_i represents the ith element in the summation. The letter i is called the **index of summation**.

Algebra of Summation: Let n be a natural number, a_i and b_i be real numbers for i from 1 to n, and $c \in \mathbb{R}$. Then

1) $\displaystyle\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$ 2) $\displaystyle\sum_{i=1}^{n} c \cdot a_i = c\sum_{i=1}^{n} a_i$

Def: A **Riemann sum** is an approximation of the area under a curve on the interval $[a, b]$ using a finite number, n, of rectangles, where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$, given by

$$A \approx \sum_{i=1}^{n} f(x_i)\ \Delta x$$

.

Def: The **exact area** under the curve $f(x)$ on $[a, b]$ is given by $A = \displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\ \Delta x$.

Formulas for Sums:

1) $\displaystyle\sum_{i=1}^{n} 1 = n$

2) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4) $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

1.1 Practice

1) Evaluate the following sums:

a) $\displaystyle\sum_{x=1}^{2} 6x$ b) $\displaystyle\sum_{i=0}^{2}(i+5)$ c) $\displaystyle\sum_{j=1}^{2} 5j$

d) $\displaystyle\sum_{l=3}^{5}(l^2-3)$ e) $\displaystyle\sum_{k=1}^{n}(2k+5)$ f) $\displaystyle\sum_{t=1}^{n} 8t^2$

2) Approximate the area under the curve $f(x) = x^2 + 2$ on the interval $[-2, 1]$ using 6 right rectangles.

3) Approximate the area under the curve $f(x) = \sqrt{x+1}$ on the interval $[-1, 0]$ using 4 left rectangles.

4) Approximate the area under the curve $f(x) = x^3 - 3$ on the interval $[1, 3]$ using 5 left rectangles.

5) Find the exact area under the curve $f(x) = 3x + 3$ from $[0, 3]$ by taking the limit of the appropriate sum.

2 Integration

2.1 Definite Integrals - Definition and Properties

Def: Let $f(x)$ be defined and continuous on the interval $[a, b]$. For each partition $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$ of the interval $[a, b]$ into n equal parts, the length of each subinterval is $\Delta x = \dfrac{b-a}{n}$ and each $x_i = a + i\Delta x$, $i = 1, 2, \ldots, n$. The **definite integral** of f on $[a, b]$ is denoted and defined by

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\ \Delta x$$

provided this limit exists. If the limit exists, we call f **integrable** on the interval $[a, b]$.

Here, a and b are called the **limits of integration**. The term $f(x)\ dx$ is a differential.

Theorem: If $f(x)$ is a continuous function on $[a, b]$, then $\displaystyle\int_a^b f(x)\ dx$ exists.

Properties of the Definite Integral: Let a, b and c be real numbers with $a < b$ and f and g be continuous functions. Then

1) $\displaystyle\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$ (since Δx is opposite sign)

2) $\displaystyle\int_a^a f(x)\ dx = 0$

3) $\displaystyle\int_a^b c\ dx = c(b-a)$

4) $\displaystyle\int_a^b (f(x) \pm g(x))\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$

5) $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$

6) $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$

7) If $f(x) \geq 0$ for all $x \in [a,b]$, then $\int_a^b f(x)\ dx \geq 0$.

8) If $f(x) \geq g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx$.

9) If $m \leq f(x) \leq M$ for all $x \in [a,b]$, then

$$m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a).$$

We can find the definite integral of a function with a finite number of jump discontinuities. If a function f is continuous on $[a,b]$ except at points $c_1, c_2, \ldots, c_n \in [a,b]$, then

$$\int_a^b f(x)\ dx = \int_a^{c_1} f(x)\ dx + \int_{c_1}^{c_2} f(x)\ dx + \cdots + \int_{c_n}^b f(x)\ dx.$$

2.2 Fundamental Theorem of Calculus

Fundamental Theorem of Calculus: Let f be continuous on $[a,b]$.

Part 1: The function A defined by

$$A(x) = \int_a^x f(t)\ dt$$

for all $x \in [a,b]$ is an antiderivative of f on $[a,b]$.

Part 2: If F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\ dx = F(x)\Big|_a^b = F(b) - F(a).$$

Combining Parts 1 and 2, we see that

$$\begin{aligned} A(x) &= \int_a^x f(t)\ dt \\ &= F(x) - F(a) \\ A'(x) &= F'(x) - 0 \\ &= f(x) \end{aligned}$$

2.3 Substitution

Recall that if $H(x) = F(g(x))$, then

$$H'(x) = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$$

where $F' = f$. Then we know that

$$\int f(g(x)) \cdot g'(x)\ dx = F(g(x)) + C.$$

We can simplify the problem by letting $u = g(x)$. Recall that we said in Section 3.6 if $y = f(x)$, then $dy = f'(x)dx$. Then here we have $du = g'(x)dx$ and

$$\int f(g(x)) \cdot g'(x)\ dx = \int f(u)\ du = F(u) + C = F(g(x)) + C.$$

Using this change of variables is called the **substitution method**.

We can also use substitution on definite integrals. Let $u = g(x)$.

$$\int_a^b f(g(x)) \cdot g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du.$$

2.4 Integration by Parts

Recall that $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$. We can rearrange this equation as follows:

$$f(x)g'(x) = (fg)'(x) - f'(x)g(x)$$

Integrating both sides, we get

$$\begin{aligned}\int f(x)g'(x)\ dx &= \int (fg)'(x)\ dx - \int f'(x)g(x)\ dx \\ &= f(x)g(x) - \int f'(x)g(x)\ dx\end{aligned}$$

Using substitution, we can make this look simpler. Let $u = f(x)$ and $v = g(x)$.

$$\int u\ dv = uv - \int v\ du$$

The mnemonic ETAIL gives an increasing difficulty in integration (choosing dv):

1) Exponential
2) Trig
3) Algebraic
4) Inverse Trig
5) Logarithmic

Try to pick an easier type to integrate, like exponential or trigonometric, if possible.

2.5 Finding Derivatives with Fundamental Theorem of Calculus

Find the derivative of the following functions using FTOC.

a) $f(x) = \int_1^x t^2\,dt$

b) $g(x) = \int_{-2}^x \sin(t)\,dt$

c) $h(x) = \int_0^x e^t\,dt$

d) $f(x) = \int_1^x \frac{1}{t^2}\,dt$

e) $f(x) = \int_1^{x^2} t^2\,dt$

f) $g(x) = \int_{-2}^{x^5} \sin(t)\,dt$

g) $h(x) = \int_0^{\ln(x)} e^t\,dt$

h) $f(x) = \int_1^{2x^3} \frac{1}{t^2}\,dt$

2.6 Indefinite Integral Practice

Integrate using any appropriate method.

a) $\int 8e^{4x}(e^{4x}-4)^{1/5}\,dx$

b) $\int \frac{1}{\sqrt{25-x^2}}\,dx$

c) $\int \frac{1}{3x+1}\,dx$

d) $\int \frac{50x}{\sec(5x^2+5)}\,dx$

e) $\int \sin^2(3x)\cos(3x)\,dx$

f) $\int x^4\sin(x)\,dx$

g) $\int \arccos(x)\,dx$

h) $\int x\sqrt{2x-1}\,dx$

i) $\int \sqrt{x}\ln(x)\,dx$

j) $\int(-9x^2+10x)\,dx$

k) $\int \frac{1}{x\sqrt{x^2-81}}\,dx$

l) $\int \ln(x+3)\,dx$

m) $\int \sqrt{2x-1}\,dx$

n) $\int 9\sin(3x)\,dx$

o) $\int x^4e^x\,dx$

p) $\int x\sec^2(x)\,dx$

q) $\int e^{-x}\tan(e^{-x})\,dx$

r) $\int 4\sin\left(\frac{x}{3}\right)\,dx$

s) $\int -5\cos(\pi x)\,dx$

t) $\int \frac{1}{16+x^2}\,dx$

u) $\int 4\tan(4x)\sec^5(4x)\,dx$

v) $\int x\cdot 2^x\,dx$

w) $\int x^2\cos(3x)\,dx$

x) $\int \frac{x}{\sqrt{25-x^2}}\,dx$

2.7 Definite Integral Practice

Integrate using any appropriate method.

a) $\displaystyle\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$

b) $\displaystyle\int_0^1 \frac{10x^4}{9+4x^{10}}\,dx$

c) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx$

d) $\displaystyle\int_1^2 \frac{\ln(x)}{x^2}\,dx$

e) $\displaystyle\int_0^{\pi/2} \frac{20x^3}{\sqrt{25-25x^8}}\,dx$

f) $\displaystyle\int_{-3}^{-2} 5\sqrt[3]{2x+4}\,dx$

g) $\displaystyle\int_0^4 \frac{1}{\sqrt{2x-1}}\,dx$

h) $\displaystyle\int_{-1}^2 \frac{2}{(2x+4)^3}\,dx$

i) $\displaystyle\int_{-1}^1 x(x^2+1)^3\,dx$

j) $\displaystyle\int_0^{\pi/2} x\cos(x)\,dx$

k) $\displaystyle\int_1^9 (x^{3/2}+2x+3)\,dx$

l) $\displaystyle\int_0^{\pi} 20x\sin(5x^2-3)\,dx$

m) $\displaystyle\int_1^2 \frac{2}{x(\ln(4x)-1)}\,dx$

n) $\displaystyle\int_4^9 \frac{1}{3\sqrt{x}+\sqrt{x}}\,dx$

o) $\displaystyle\int_0^1 \frac{20e^{5x}}{e^{5x}+3}\,dx$

p) $\displaystyle\int_0^{\pi/2} x\sin(2x)\,dx$

q) $\displaystyle\int_{-2}^1 (2t^2-1)^2\,dt$

r) $\displaystyle\int_{-1}^1 e^{2x-2}\,dx$

s) $\displaystyle\int_{-4}^{-2} (|-3x-9|-x)\,dx$

t) $\displaystyle\int_{-5}^1 -|x^2+4x|\,dx$

u) $\displaystyle\int_2^4 x^2e^{2x}\,dx$

v) $\displaystyle\int_{-1}^0 18x^2(3x^3+3)^2\,dx$

w) $\displaystyle\int_{-2}^{-1} \frac{4}{x^2}\,dx$

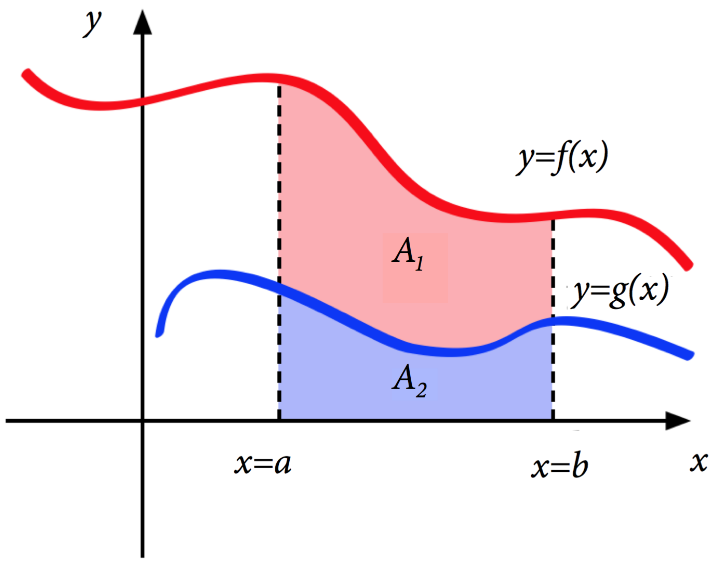
x) $\displaystyle\int_0^{\pi} \frac{-4\sin(4x)}{\sqrt{9-\cos^2(4x)}}\,dx$

y) $\displaystyle\int_0^1 \frac{24x}{(4x^2+4)^2}\,dx$

z) $\displaystyle\int_0^5 xe^{-x}\,dx$

1 Area Between Curves

In Chapter 4, we developed definite integrals to calculate the area "under" a curve - between the graph of a function and the x-axis. We may want to find the area of a more complicated shape, such as the area between two curves.



From the picture, we can see that the area under the curve $y = f(x)$ is given by

$$\int_a^b f(x)dx = A_1 + A_2$$

while the area under the curve $y = g(x)$ is given by

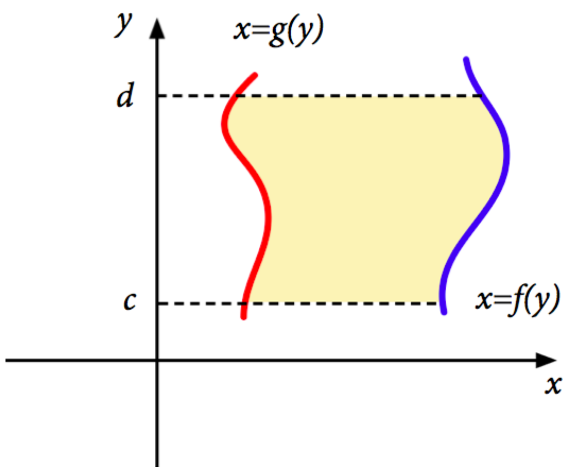
$$\int_a^b g(x)dx = A_2.$$

If we wanted to find just the area between the two curves, A_1, we could do the following:

$$\begin{aligned} A_1 &= (A_1 + A_2) - A_2 \\ &= \int_a^b f(x)dx - \int_a^b g(x)dx \\ &= \int_a^b (f(x) - g(x))dx \end{aligned}$$

This yields the formula we will use to find the area "between two curves".

Sometimes, it may make more sense to view an area as the horizontal distance between two curves, as in the image below.



Similar to before, we can find this area using the integral

$$\int_c^d (f(y) - g(y))dy$$

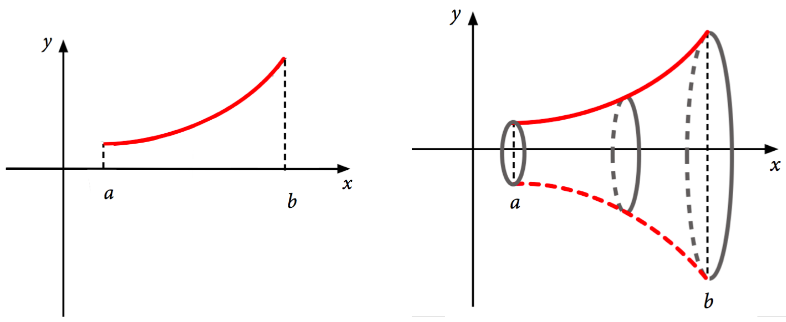
Note that here we have the "curve to the right" minus the "curve to the left".

1.1 Practice

1) Find the area of the region bounded by $y = 4x - x^2$ and $y = 3$.
2) Find the area of the region bounded by $y = x^2$ and $y = 2x^2 - 25$.
3) Find the area of the region bounded by $y = 7x - 2x^2$ and $y = 3x$.
4) Find the area of the region bounded by $y = x(x-3)(x+3)$ and $y = 7x$.
5) Find the area of the region bounded by $y = x^3 - 1$ and $y = x^2 - 1$.
6) Find the area of the region bounded by $x = 2y^2 + 12y + 19$, $x = -\frac{y^2}{2} - 4y - 10$, $y = -3$ and $y = -2$.
7) Find the area of the region bounded by $y = -2\sec^2(x)$, $y = 2\cos(x)$, $x = 0$ and $x = \frac{\pi}{4}$.
8) Find the area of the region bounded by $y = 2x^{2/3}$ and $y = x$.
9) Find the area of the region bounded by $y = -x^3 + 6x$ and $y = -x^2$.
10) Find the area of the region bounded by $y = -\frac{x^3}{2} + 2x^2$ and $y = -x^2 + 4x$.

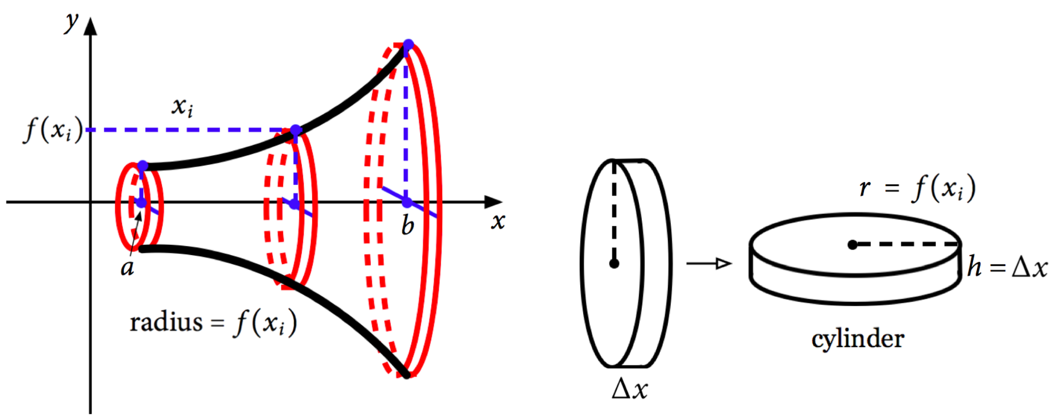
2 Volumes of Solids of Revolution

Def: A **solid of revolution** is formed by revolving a planar region around a line in the plane called its **axis of revolution**.



Method 1: Disks

For a solid of revolution with no hole in the middle, you may use the **disk method** to calculate volume.



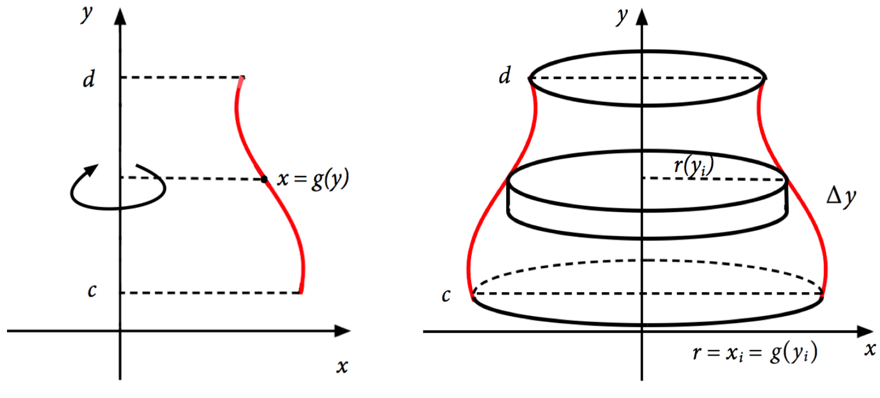
The volume of a cylinder is $V = \pi r^2 h$. Then the exact volume of this shape is

$$V = \int_a^b \pi (f(x))^2 \, dx$$

In general, if radius is $r(x)$, then

$$V = \int_a^b \pi (r(x))^2 \, dx.$$

Notice that we could also revolve a function $x = g(y)$ about the y-axis to generate a solid of revolution.

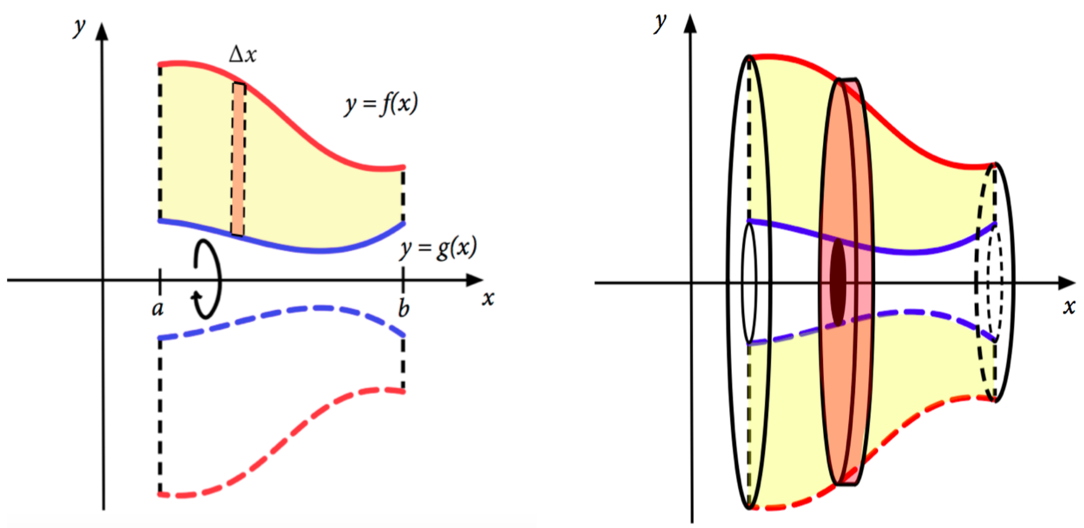


Here, the volume is given by

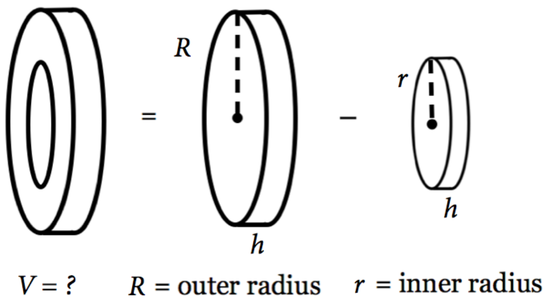
$$V = \int_c^d \pi(r(y))^2 \, dy.$$

Method 2: Washers

For a solid of revolution with a hole in the middle, we may use the **washer method**.



We need a new volume formula for a washer instead of a disk.



$$V = \pi R^2 h - \pi r^2 h$$
$$= \pi(R^2 - r^2)h$$

where R is the outer radius, r is the inner radius, and h is the thickness of the washer.

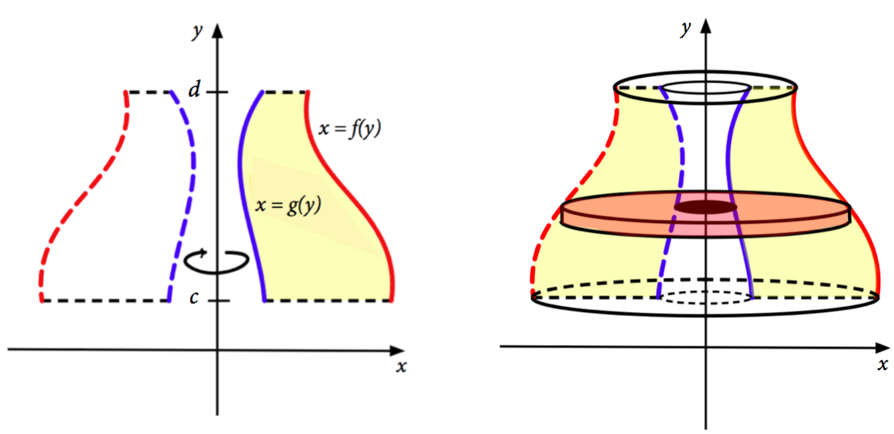
Then if $f(x)$ is the outer or "bigger" function and $g(x)$ is the inner, the volume of the solid is given by

$$V = \int_a^b \pi \left((f(x))^2 - (g(x))^2 \right) dx$$

For convenience, we say

$$V = \int_a^b \pi \left((R(x))^2 - (r(x))^2 \right) dx.$$

Similarly, we may have the area between two curves $f(y)$ and $g(y)$ revolved around the y-axis.

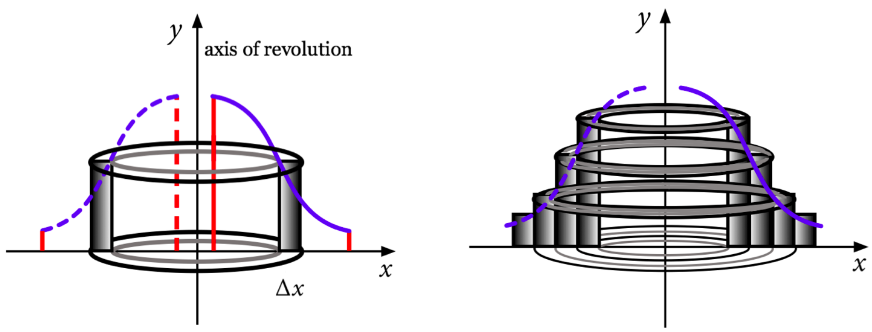


As you might expect, the volume formula here is

$$V = \int_c^d \pi \left((f(y))^2 - (g(y))^2 \right) dy = \int_c^d \pi \left((R(y))^2 - (r(y))^2 \right) dy.$$

Method 3: Shells

If we revolve the area under a curve $y = f(x)$ around the y-axis (or $x = f(y)$ around the x-axis), we may use the **cylindrical shell method**. We take a thin slice of the area, revolve it around the parallel axis, and calculate the volume of the cylinder it creates.



Then since each cylindrical shell has volume $2\pi rh$,

$$V = \int_a^b 2\pi r(x)\ h(x)dx, \quad V = \int_c^d 2\pi r(y)\ h(y)dy.$$

2.1 Practice

1) Find the volume of the solid resulting from rotating the region enclosed by $y = \sqrt{\sin(x)}$, the x-axis, $x = 0$ and $x = \pi$ around the x-axis.

2) Find the volume of the solid resulting from rotating the region enclosed by $y = x^2$, $y = 4$, and the y-axis around the y-axis.

3) Find the volume of the solid resulting from rotating the region enclosed by $y = 2 - x^2$ and $y = 1$ around the axis $y = 1$.

4) Find the volume of the solid resulting from rotating the region enclosed by $x = -y^2 + 2$ and $y = x$ around the axis $x = -2$.

5) Find the volume of the solid resulting from rotating the region enclosed by $y = \sqrt{x} + 1$ and $y = x^2 + 1$ around the axis $y = -1$.

6) Find the volume of the solid resulting from rotating the region enclosed by $y = \sqrt{x}$, $y = 0$ and $x = 4$ around the following axes:

   a) x-axis　　b) y-axis　　c) $x = 4$　　d) $x = 6$

7) Use the disk method to derive the formula for the volume of a sphere with radius r.