1 Conic Sections

1.1 Parabola

- Standard Form (Vertical):
  $y - k = a(x - h)^2$
- Vertex: (h, k)
- Focus: $\left(h, k + \frac{1}{4a}\right)$
- Directrix: $y = k - \frac{1}{4a}$

- Standard Form (Horizontal):
  $x - h = a(y - k)^2$
- Vertex: (h, k)
- Focus: $\left(h + \frac{1}{4a}, k\right)$
- Directrix: $x = h - \frac{1}{4a}$

1. Find the vertex, focus, and directrix of $y + 2 = 2(x - 3)^2$.

1.2 Ellipse

- Standard Form (Vertical): $a > b$
  $\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$
- Vertices: $(h, k \pm a)$
- Endpoints of the Minor Axis: $(h \pm b, k)$
- Foci: $(h, k \pm \sqrt{a^2 - b^2})$

- Standard Form (Horizontal): $a > b$
  $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
- Vertices: $(h \pm a, k)$
- Endpoints of the Minor Axis: $(h, k \pm b)$
- Foci: $(h \pm \sqrt{a^2 - b^2}, k)$

1. Find the vertices, endpoints of the minor axis, and foci of $\frac{(x-2)^2}{9} + \frac{(y+1)^2}{4} = 1$.

2. Find the standard form of the ellipse with vertices at $(4, 0)$ and $(-4, 0)$ and foci at $(3, 0)$ and $(-3, 0)$.

1.3 Hyperbola

- Standard Form (Vertical): $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$
- Center: (h, k)
- Vertices: $(h, k \pm a)$
- Foci: $(h, k \pm \sqrt{a^2 + b^2})$
- Asymptotes: $y = \pm \dfrac{a}{b}(x-h) + k$

- Standard Form (Horizontal): $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$
- Center: (h, k)
- Vertices: $(h \pm a, k)$
- Foci: $(h \pm \sqrt{a^2 + b^2}, k)$
- Asymptotes: $y = \pm \dfrac{b}{a}(x-h) + k$

1. Find the standard form of the hyperbola with vertices at $(0, -2)$ and $(6, -2)$ and foci at $(-2, -2)$ and $(8, -2)$.

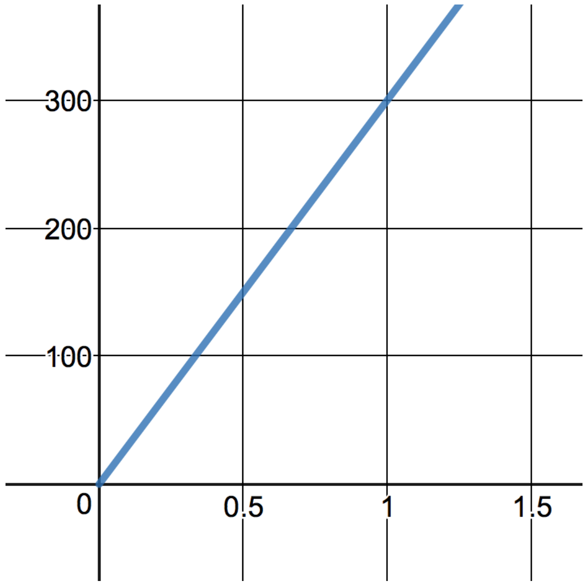
1.4 Determining Type

1. Do the following equations represent a parabola, an ellipse, or a hyperbola?
   (a) $2x^2 + 4x + y^2 - 6y - 13 = 0$
   (b) $4x^2 - 8x - 3y^2 - 6y - 11 = 0$
   (c) $x + 5y^2 + 20y + 19 = 0$

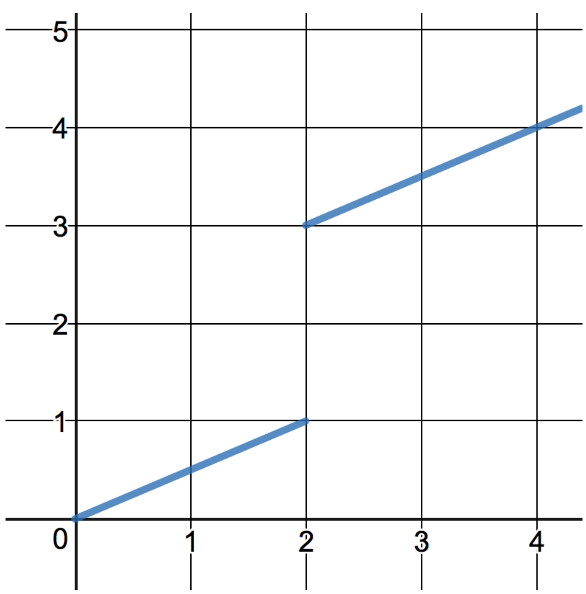
2. Can you answer the previous question without finding the standard form?

1 Interpreting Graphs

In calculus, we frequently discuss functions related to the movement of an object. A function measuring distance is labeled as $s(t)$, position s as time t changes. Consider the following two graphs.



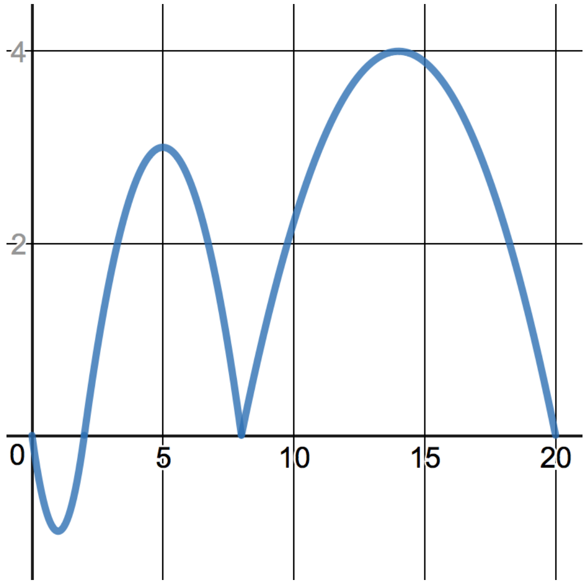
In Graph 1, the horizontal axis is time (in seconds), while the vertical axis is distance from starting position (in miles).



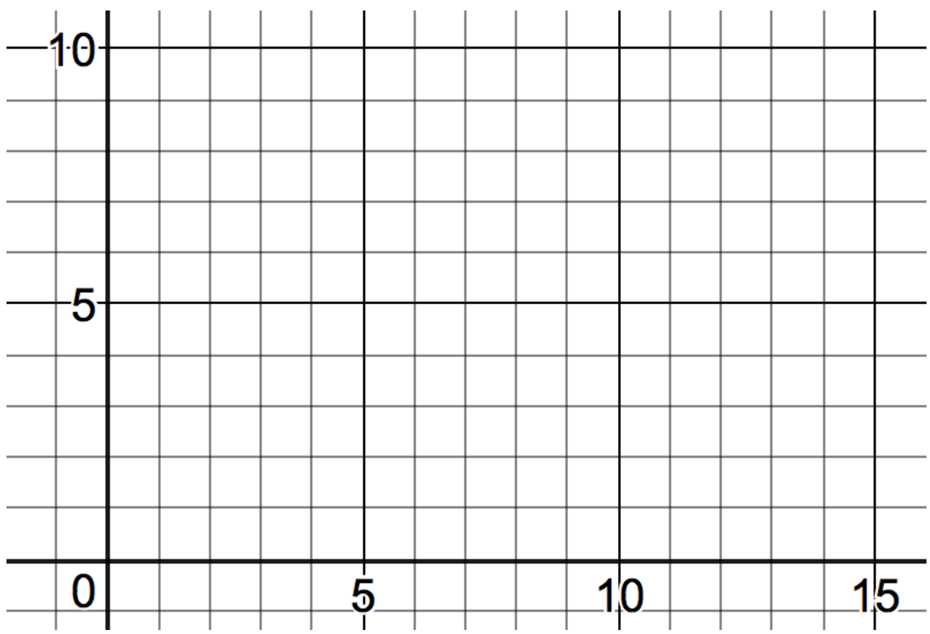
In Graph 2, the horizontal axis is time (in seconds), while the vertical axis is distance from starting position (in feet).

1. One of these graphs represents the movements of the Flash, and one represents the movements of Dr. Strange. Which is which? [Alternatively: what superpower is represented in each of these graphs?] Analyze the information about distance over time in order to determine the answer.

2. The below graph is the velocity of your car on your drive from home to NC State. Describe the car's movements based on the graph. The graph starts at the origin (home), when $t = 0$ and speed is 0.

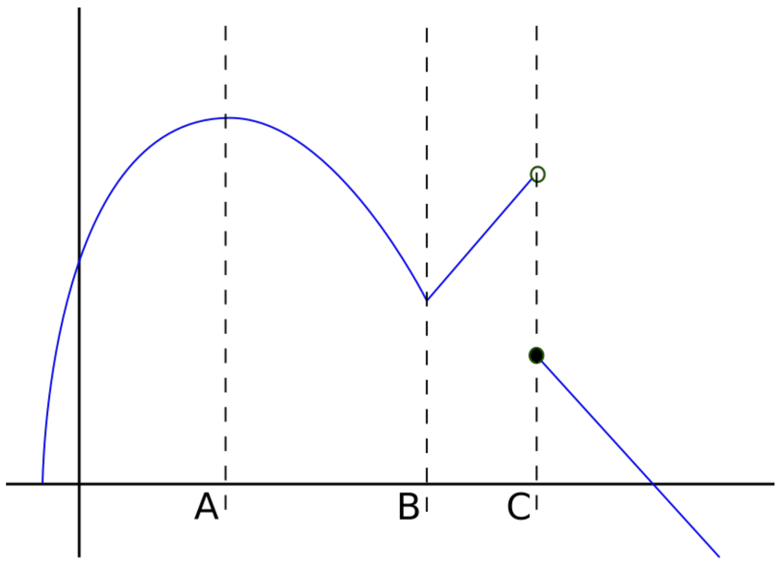


3. Draw a velocity graph that represents the following 15 minute run:
   (a) Runs at a constant pace for the first 5 minutes
   (b) Slows down to a stop over the next 5 minutes
   (c) Stops for 2 minutes
   (d) Speeds back up to their original pace over the next 3 minutes.



1 Exploring Limits and Continuity

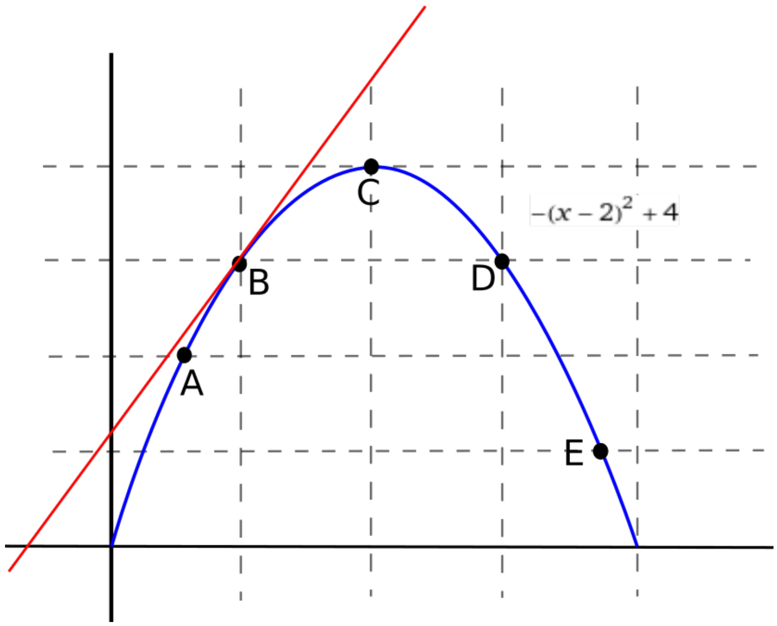
1. For the following graph, answer the question for $x = A$, $x = B$ and $x = C$.



   (a) Does the limit of the function exist at each input?

   (b) Is the function continuous at each input?

2. For the following graph of the function $f(x) = -(x-2)^2 + 4$, answer the questions:



(a) Which point, A, C, D or E, yields the secant line through B with the best approximation of the slope of the tangent line at B?

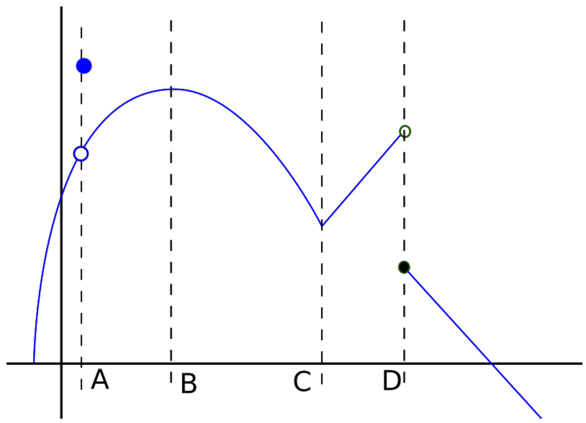
(b) What is the slope of the tangent line at B, if $x = 1$?

(c) What is the slope of the tangent line at C, if $x = 2$?

1 Limits and Continuity

1. For $f(x)$ defined below and $A = 2$, $B = 4$, $C = 8$ and $D = 10$ we have the following picture.

$$f(x) = \begin{cases} 12 & \text{for } x = 2 \\ -\frac{1}{4}(x-4)^2 + 10 & \text{for } x < 8, x \neq 2 \\ x - 2 & \text{for } 8 \leq x < 10 \\ -\frac{2}{3}(x-16) & \text{for } x \geq 10 \end{cases}$$



(a) What is the limit of the function at $x = 2, x = 4, x = 8$ and $x = 10$?

(b) Is the function continuous at $x = 2$, $x = 4$, $x = 8$ or $x = 10$? If not, which continuity criteria does it fail?

2. Consider each function. Where is it continuous?

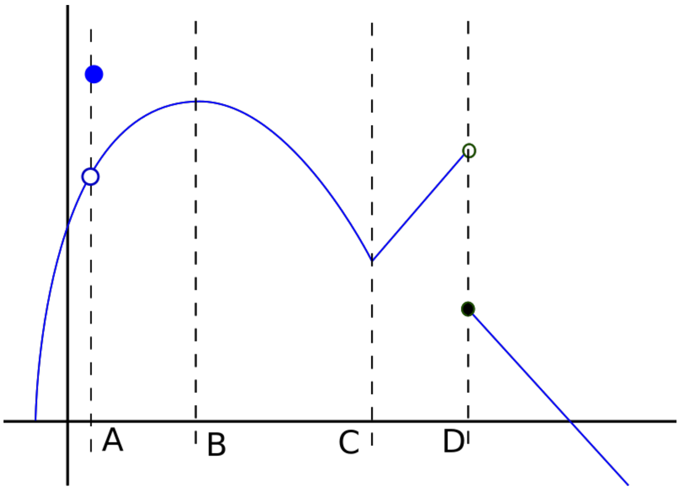
   (a) $f(x) = \sqrt{x}$ (b) $f(x) = e^x$ (c) $f(x) = \dfrac{2x-1}{x}$

3. What is the difference between "continuous" and "continuous on its domain"?

4. The vertical height of a football, kicked from the ground, is given by the function $s(t) = -3t^2 + 18t$, where t is in seconds and s is in yards. What is the velocity of the football when it hits the ground? In this context, what does negative velocity mean?

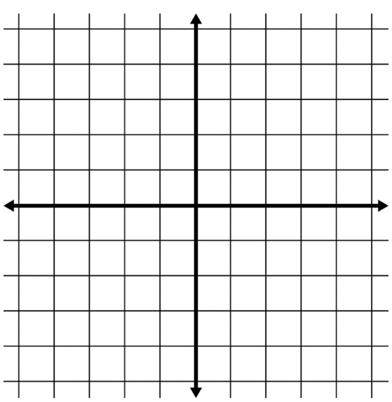
1 Derivatives

1. For the graph below, $f(x) = \begin{cases} 12 & \text{for } x = 2 \\ -\frac{1}{4}(x-4)^2 + 10 & \text{for } x < 8, x \neq 2 \\ x - 2 & \text{for } 8 \leq x < 10 \\ -\frac{2}{3}(x-16) & \text{for } x \geq 10 \end{cases}$.



Is f differentiable at $A = 2$, $B = 4$, $C = 8$ or $D = 10$? (Note that we can answer by sight for A, C and D.)

2. Let $f(x) = x^2$. Find $f'(-2), f'(0)$ and $f'(1)$. Now draw the graph of f and sketch in tangent lines at $x = -2, x = 0$ and $x = 1$. How do the tangent lines relate to the derivatives you calculated?



3. Let $f(x) = \dfrac{1}{x-2}$. Find $f'(0)$ and $f'(2)$.

4. Using one-sided derivatives, show that $f(x) = |x|$ is not differentiable at $x = 0$. (Hint: It may help to think of $|x|$ as a piecewise function.)

5. Water is being pumped into a tank such that the volume V (in gallons) at time t (in minutes) is given by $V(t) = 75t^3$. Find the average rate of change of V with respect to time from $t = 1$ min to $t = 2$ min. Then find the instantaneous rate of change of V with respect to t at $t = 1$ min, i.e. find the derivative of $V(t)$ with respect to t when $t = 1$ min.

6. When an object is dropped from a height of 100 feet, its position above the ground t seconds after being dropped is given by $s(t) = 100 - 16t^2$. Find its velocity and acceleration as functions of time. What is the object's velocity and acceleration when it hits the ground?

1 Derivatives

1. Differentiate the following functions:

   (a) $f(x) = (-x^2 + 6x - 4)^5$

   (b) $f(x) = \sin(4x^2 + 3x)$

   (c) $f(x) = \sqrt[4]{-3x^4 - 2}$

   (d) $f(x) = (3x^3 + 1)(-4x^2 - 3)^4$

   (e) $f(x) = \dfrac{-2x^2 - 5}{\cos(2x^3)}$

   (f) $f(x) = \cos[(-3x^2 + 2)^2]$

   (g) $f(x) = \sin^3(x^5)$

   (h) $f(x) = ((x + 5)^5 - 1)^4$

   (i) $f(x) = (5x^3 - 3)^5 \sqrt[4]{-4x^5 - 3}$

   (j) $f(x) = \dfrac{(x^3 + 4)^5}{3x^4 - 2}$

2. Prove that the derivative of $f(x) = \csc(x)$ is $f'(x) = -\csc(x)\cot(x)$.

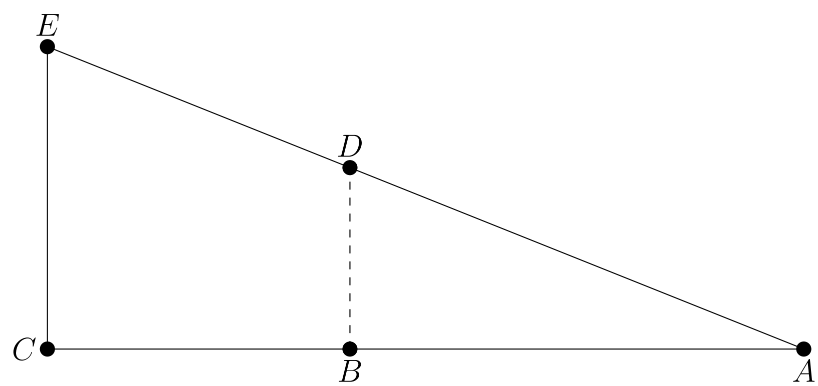
3. Come up with a function that requires three applications of the chain rule to differentiate. Then differentiate it.

1 Logarithmic Differentiation

Use logarithmic differentiation to find the derivative of $f(x) = x^{\sin x}$.

2 Related Rates

1. A streetlight is on a pole 14 ft tall. A boy 4 ft tall walks away from the pole at a rate of 5 ft/sec. At what rate is his shadow lengthening when he is 25 ft from the pole? At what rate (in feet per second) is the tip of his shadow moving?

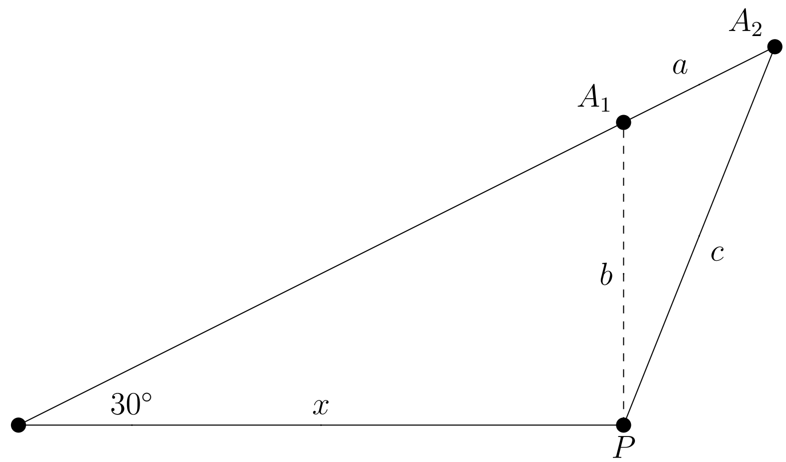


Let B be where the boy is standing and C be the base of the lamppost. Then AEC and ABD are similar triangles and the side lengths are in the same proportions. E.g.

$$\frac{\overline{AB}}{\overline{BD}} = \frac{\overline{AC}}{\overline{CE}}.$$

Now, let $s = \overline{BA}$, $x = \overline{CB}$ and $L = \overline{AC}$. Then the rate that the shadow is lengthening is $\dfrac{ds}{dt}$, and the rate the tip of the shadow is moving is $\dfrac{dL}{dt}$.

2. An airplane, flying at a constant speed of 360 mi/hr and climbing at a 30° angle, passes over a person, P, on the ground when it is at an altitude of 3 mi, at point A_1. Find the rate of change, in mi/hr, of the distance from the person on the ground to the plane one minute later, when the plane is at A_2.



Let's look at the triangle formed by P, A_1, and A_2. We can relate the sides a, b and c using Law of Cosines if we can find the measurement of one of the angles.

What is the measurement of $\angle PA_1A_2$?

Since we can find the measurement of $\angle PA_1A_2$, we use the following version of Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab\cos(\angle PA_1A_2)$$

After implicit differentiation, we will need a_0, b_0 and c_0 for the moment when the plane is one minute past it's position above the person. We also need $\frac{da}{dt}$ and $\frac{db}{dt}$. We are trying to find $\frac{dc}{dt}$.

What are $\frac{da}{dt}$ and $\frac{db}{dt}$? How do we find a_0, b_0 and c_0?

1 Newton's Method

Newton's method to approximate the root x^* of a function f is the iterative process, with initial guess x_1,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

1. We sometimes use Newton's Method to find the smallest positive root of a given function. What is a smart initial guess for this type of problem?
2. Newton's Method will fail for $f(x) = \cos x$ when the first guess is $x_1 = 0$. Why? (Hint: try to find x_2.) What is special about the point $x = 0$ on this graph? What if we make our initial guess $x_1 = 1.5$? How many iterations do we need to do before our answers match to 2 decimal places?
3. What happens if we try to find a root of $f(x) = x^3 - 3x^2 + x - 1$ with initial guess $x_1 = 1$? Calculate up to x_3 and describe what is happening. Now try it with $x_1 = 3$ and calculate up to x_5. What might you conclude about f from the second calculation?

2 Extreme Values

1. Find all relative maxima and minima for the following functions.
   (a) $y = -\dfrac{2}{x^2 - 4}$
   (b) $f(x) = -\dfrac{1}{5}(x-4)^{5/3} - 2(x-4)^{2/3}$

2. The position of a particle moving along the x-axis is given by the function $s(t) = t^4 - 6t^3 + 12t^2 - 108t$, for $t \in [0, 8]$.
   (a) Find the velocity and acceleration functions.
   (b) When is the object moving to the left? To the right? When is it stationary?
   (c) What is the position of the particle when it is the farthest left? When does this occur? What is the farthest right it will go?
   (d) What is the maximum velocity of the particle?

3. For $g(x) = x^4 - x^2$, find the following:
   (a) When is g increasing/decreasing?
   (b) What are the relative extrema of g?
   (c) When is g concave up/down?
   (d) What are the point(s) of inflection of g?

1 Optimization

1. When ticket price is \$40, the average attendance at a football game is $40,000$. For every decrease of \$1 in ticket price, an additional $2,000$ people will attend. What price should be charged per ticket to maximize revenue? How many fans will attend at this price? What is the maximum revenue?

2. A piece of wire 14 ft long is cut into two pieces. One piece is used to make a square and the other is used to make a circle. Where should the wire be cut so that the combined area of the two shapes is a maximum? Where should it be cut so that the combined area is a minimum?

1 Differentials

1. If $f(x) = 2\cos^2(x)$, compute its differential df and approximate the change in f when x changes from $x = \dfrac{\pi}{6}$ to $x = \dfrac{\pi}{6} + 0.1$. Approximate the relative change in f as x undergoes this change.

2. Use differentials to approximate the increase in volume of a cube if the length of each edge is changed from 10 cm to 10.1 cm. What is the exact change in volume?

2 Antiderivatives

Find the following antiderivatives:

1. $\displaystyle\int \big(x^3 - 5\sin(x)\big)\,dx$

2. $\displaystyle\int \big(x^5 - 2x^3 + x - 2\big)\,dx$

3. $\displaystyle\int \left(5e^x + \frac{3}{x}\right)dx$

4. $\displaystyle\int \frac{2}{1+x^2}\,dx$

5. $\displaystyle\int 8\sec(2x)\tan(2x)\,dx$

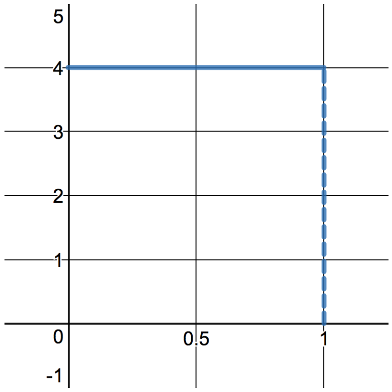
3 Areas

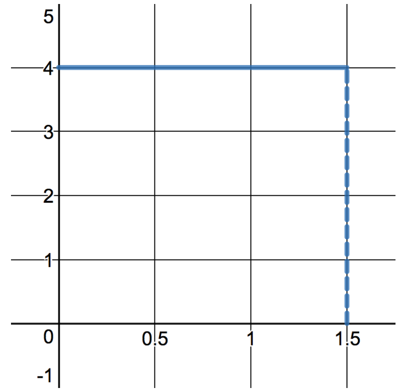
1. Use geometry to find the area between $y = 2x + 2$ and the x-axis on the interval $[0, 2]$.

2. Use the limit definition of area to compute the area between the curve $y = 16 - x^2$ and the x-axis on $[1, 3]$.

1 Thinking about Position and Velocity

Imagine you are out for a jog and are running at a consistent pace of 4 mph.





1. How far have you run after 1 hour? After 1.5 hours?
2. Use geometry to find the area under the line $y = 4$ and above the x-axis on the interval $[0, 1]$ and then $[0, 1.5]$. How do your answers relate to the previous question?
3. If your speed is 4 mph, what is your position function, $s(t)$?
4. What is the differential of your position function, $s(t)$?
5. In this example, there is no error in using the differential to calculate distance travelled. Why?

2 Velocity as Accumulation of Distance

Imagine you are out for a jog and measure your speed at evenly spaced intervals during your run. Let's use the velocity information to talk about the distance covered.

1. You check your speed at the following points:

$t = 10\ mins$	$v = 3\ mph$
$t = 20\ mins$	$v = 3.5\ mph$
$t = 30\ mins$	$v = 4\ mph$
$t = 40\ mins$	$v = 4\ mph$
$t = 50\ mins$	$v = 3.75\ mph$
$t = 60\ mins$	$v = 3\ mph$

   How can we use this information to approximate distance covered in 60 $mins$? (Hint: Think about differentials / using rectangles to approximate area.)

2. What happens to our estimate if we check speed more frequently? Every 5 $mins$? Every 1 min? Every 1 sec?

3 Summarizing Fundamental Theorem of Calculus

1. The second part of the fundamental theorem of calculus states that $\int_a^b f(x)dx = F(b) - F(a)$ for ANY antiderivative $F(x)$ of $f(x)$.

   (a) Consider the function $f(x) = \frac{x^2}{3}$. Find an antiderivative $F(x)$ of $f(x)$.

   (b) Consider the antiderivative with the constant $C = 5$. Use the fundamental theorem of calculus to compute $\int_a^b f(x)dx$.

   (c) Now consider $F(x)$ with the constant $C = 10$ and again use the fundamental theorem of calculus to compute $\int_a^b f(x)dx$.

   (d) So, why does $\int_a^b f(x)dx = F(b) - F(a)$ hold for ANY antiderivative $F(x)$?

   (e) Compute the following definite integrals:

   i. $\int_1^3 (x^3 - 2x + 1)\ dx$

   ii. $\int_0^{\pi/2} \sin(\theta)\, d\theta$

   iii. $\int_1^3 dx$

   iv. $\int_1^{\ln(5)} e^x\ dx$

2. The first part of the fundamental theorem of calculus states that if $G(x) = \int_a^x f(t)dt$ then $G'(x) = f(x)$.

   (a) Let $F(x)$ be an antiderivative of $f(x)$ and use the second part of the fundamental theorem of calculus to compute $\int_a^x f(t)\ dt$ in terms of F.

   (b) Now using what you found above, compute the derivative of $G(x) = \int_a^x f(t)\ dt$ in terms of F.

   (c) Now, what can we say about $G'(x)$?

   (d) Find the derivatives of the following functions:

   i. $G(x) = \int_1^x \sec^2(t)\ dt$

   ii. $H(x) = \int_1^x \frac{1}{\sqrt{1-t^2}}\ dt$

   iii. $H(x) = \int_1^{3x^2} \sec^2(t)\ dt$

   iv. $G(x) = \int_1^{e^x} \ln(t)\ dt$

1 Substitution Method

1.1 Indefinite Integrals

Find the following antiderivatives:

1. $\displaystyle\int e^x \cos(e^x)\, dx$

2. $\displaystyle\int (5t^2+1)(5t^3+3t-1)^{1/4}\, dt$

3. $\displaystyle\int \frac{1}{\sqrt{1-16x^2}}\, dx$

4. $\displaystyle\int \sec^4(x)\, dx$

1.2 Definite Integrals

Evaluate the following definite integrals:

1. $\displaystyle\int_{-1}^{1} x^2(3x^3-2)^4\,dx$

2. $\displaystyle\int_{0}^{\pi/4} \cos^3(4x)\sin(4x)\,dx$

1 Integration by Parts

1.1 Indefinite Integrals

Find the following antiderivatives:

1. $\displaystyle\int x\cos(x)\,dx$

2. $\displaystyle\int x\cdot 4^x\,dx$

3. $\displaystyle\int \arcsin(x)\,dx$

1.2 Definite Integrals

Evaluate the following definite integrals:

1. $\displaystyle\int_1^e x^2 \ln(x)\, dx$

2. $\displaystyle\int_1^e \ln(x)\ dx$

1 Area Between Two Curves

1. Find the area of the region bounded by the curves $y = 2x^2 - x - 4$, $y = x^3 + 2$, $x = -1$ and $x = 1$.

2. Find the area of the region bounded by the curves $y = 5x^2 - 19$ and $y = -x^2 + 5$.

3. Two motorcycles leave Raleigh at the same time. The velocity of motorcycle A is given by $v_A(t) = 100t - 50t^2$ mph and the velocity of motorcycle B is given by $v_B(t) = 50t - 25t^2$ mph. If time $t = 0$ is when they leave Raleigh, show that both motorcycles have come to a stop after two hours and that motorcycle A went farther than motorcycle B. Finally, find how much farther A went than B.